SRI VENKATESWARA INTERNSHIP PROGRAM FOR RESEARCH IN ACADEMICS (SRI-VIPRA)

Project Report of 2023: SVP - 2303

"Exploring Chaos and its Related Properties in Topological Dynamical Systems"

IQAC
Sri Venkateswara College
University of Delhi
Benito Juarez Marg Road, New Delhi
New Delhi - 110021

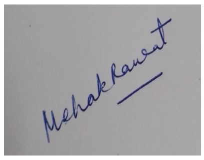
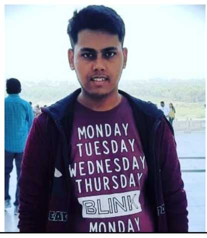
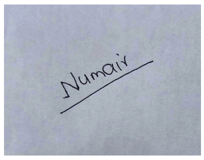
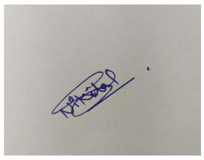
SRIVIPRA PROJECT 2023

Title : Exploring Chaos and its Related Properties in Topological Dynamical Systems

• Name of Mentor: Dr. Rahul Thakur • Name of Department: Mathematics • Designation: Assistant Professor	

Student under the SRIVIPRA Project

S.No	Photo	Name of the Student	Roll No	Course	Signature
1		Mr. Abhay Palande	1721008	B.Sc.(H) Maths Semester: IV	
2		Mr. Advitya Bhardwaj	1720108	B.Sc.(H) Maths Semester: VI	

3		Ms. Ankita Malik	1821004	B.Sc.(H) Physics Semester: IV	
4		Ms. Mehak Rawat	1721029	B.Sc.(H) Maths Semester: IV	
5		Mr. Mohammad Numair	1721023	B.Sc.(H) Maths Semester: IV	
6		Ms. Nikita Agrawal	1721067	B.Sc.(H) Maths Semester: IV	

Rahul Thakur

Signature of Mentor

Certificate

This is to certify that the aforementioned students from Sri Venkateswara College have participated in the summer project SVP - 2303 titled **"Exploring Chaos and its Related Properties in Topological Dynamical Systems"**. The participants have carried out the research project work under my guidance and supervision from 15 June 2023 to 15 September 2023. The work carried out is a systematic review of the literature available on the above mentioned subject and was done in a hybrid mode.

Rahul Thakur

Signature of Mentor

Acknowledgements

We would like to place on record our sincere thanks to the Mathematics Department, Sri Venkateswara College, University of Delhi for allowing the students to take up a research project. We would also like to express our gratitude towards **Dr. Rahul Thakur** for his constant support, encouragement, guidance, and suggestions throughout this internship. Lastly, we thank the college for coming up with this initiative and giving a platform to the budding mathematicians.

15-09-2023

Abhay Raj Palande Semester: IV Roll No: 1721008

Advitya Bhardwaj Semester: VI Roll no. 1720108

Ankita Malik Semester: IV Roll No: 1821004

Mehak Rawat Semester: IV Roll no. 1721029

Mohammad Numair Semester: IV Roll No: 1721023

Nikita Agrawal Semester: IV Roll No: 1721067

Contents

List of Symbols

The below given list describes several symbols that will be later used within the body of the document :

Number sets and logic notations

$\mathbb{N}$	Natural Numbers
$\notin$	is not a member of
$\overline{A}$	Closure of set A
$\exists$	There exists at least one
$\forall$	For all
$\Rightarrow$	Implies
$\setminus$	Set Difference
$\cap$	Set Intersection
$\cup$	Set Union
$\varnothing$	Empty Set
$\in$	is member of
$\mathbb{C}$	Complex numbers
$\mathbb{Q}$	Rational Numbers

$\mathbb{R}$ Real numbers

$\mathbb{Z}$ Integers

$\subset$ is a proper subset of

$\subseteq$ is a subset of

Other symbols

inf Infimum or Minimum

lim Limit

$\liminf$ Limit Inferior

$\limsup$ Limit Superior

sup Supremum or Maximum

∞ Infinity

Relation Operators

$\neq$ Not equal to

$<$ less than

$=$ equal to

$>$ greater than

$\geqslant$ greater than or equal to

$\leqslant$ less than or equal to

Preface

The project entitled **"Exploring Chaos and its Related Properties in Topological Dynamical Systems"** comprises five chapters. At the beginning of each chapter we give a brief outline of the research work carried out in that chapter. The report is organized as follows: **Chapter 1** is the introductory chapter of the section. It discusses the significance and motivation of each topic in brief. In **Chapter 2**, we discuss the Transitivity and Density of Periodic Points. In **Chapter 3**, we study Sensitivity and its Stronger Forms. We also discuss various interesting examples that further our understanding of sensitivity in the context of Chaos. In **Chapter 4**, we discuss Chaos Theory and the various types of Chaos. In **Chapter 5**, we discuss the Product of Dynamical Systems and the various theorems associated with them.

Chapter 1

Introduction

"A butterfly's wings might create tiny changes in the atmosphere and ultimately become the decisive factor that causes a tornado"

Edward Norton Lorenz

The name "Chaos" (*Greek* , translit. Cháos) is presumably derived from the Greek verbs (cháskō) and (chaínō), both meaning "gape, be wide open," and both themselves related to the Proto-Indo-European $'g^h eh_2 n$, "gape." [7]

The earliest reference to the idea of Chaos in *Western Culture* is found in Hesiod's Theogony, where it is described simply as the first entity that came into existence. [8]

The earliest reference to the idea of Chaos in *Indian Culture* is found in Vedic Literature, which recounts the multiple attempts of Prajāpati to create the universe. In one such attempt, he creates a universe that is too fragmented or chaotic(prthāk).When entities in the universe are too individualist, separated, or different from each other — prthāk, they cannot connect hence chaotic. [9]

Earliest known *mathematical* evidence of Chaos Theory is through the works of 18th-century French Mathematician Pierre-Simon Laplace. He demonstrated that the totality of celestial body motions (at his time, the sun and the planets) could be explained by the laws of Newton, reducing the study of planets to a series of differential equations. But there was a catch. His calculations depended on the capacity to know the initial conditions of the system, an unusual challenge for mathematicians of that time who were used to determinism. The phenomenon of sensitivity to initial conditions was discovered by Poincaré in his study of the n-body problem [10].

Edward Lorenz, from the Massachusetts Institute of Technology (MIT), USA is the official discoverer of *Chaos Theory* [11]. He first observed the phenomenon in the early 1960s. while making calculations with uncontrolled approximations aiming at predicting the weather by making the same calculation rounding with 3- 3-digit rather than 6-digit numbers which did not provide the same solutions.

It is well known that in nonlinear systems, multiplications during iterative processes amplify differences in an exponential manner. It is in this experiment, that we also observe the modern-day ideas concerning Sensitivity taking shape as well. In the 1970s, French-American mathematician Benoit Mandelbrot discovered Fractals and Mandelbrot Sets [12].

Chaos refers to the complex and unpredictable behavior of some dynamical systems. Although these systems are deterministic, even a small change in the initial conditions can lead to a large change in the behavior of the system over time. The first topological definition of chaos was introduced by Li and Yorke. R. L. Devaney's mathematical definition identifies three components of a chaotic map: transitivity, sensitivity to initial conditions, and density of periodic points [1].

1.1 Transitivity and Density of Periodic Points

Auslander and Yorke first associated transitivity with the definition of chaos. A transitive function f is one in which any two open sets can be connected by a chain of iterates of the function. This is necessary because if there are such sets that cannot be connected, then some regions will be inaccessible from other parts of the system, and therefore the system will be predictable in those regions. Thus, the system is indecomposable, i.e., it cannot be broken down into subsystems that do not interact under f.

Then, the chaotic system has elements of unpredictability as well as regularity, i.e., sensitivity and density of periodic points respectively. Sensitivity means that the orbits of arbitrarily close points diverge. The density of periodic points means that every open set has points that are predictable. Devaney incorporated this idea into the definition of chaos, which along with transitivity implies chaos on infinite sets as proved by Banks et al.

Even the density of periodic points on intervals is redundant. Also, the existence of a dense orbit is a stronger condition that implies transitivity (if the space has no isolated points). For some special spaces, the converse is also true. Finally, in a special way chaos exists on unit interval given the density of points with period three and higher.

We define a few terms that we will be using in this section and briefly introduce their contents.

Definitions

Note: For easier understanding to the reader examples have also been given with each definition.

Let X be a Topological Space and f be a continuous mapping defined on X , then:

Definition 1.1.1 (Semiflow)**.** Let Y be a metric space and T be an abelian topological monoid action, then a *semiflow*; denoted by (Y, T) is a left monoid action on Y.

Essentially, a semiflow denotes the action of time, T on a metric space, Y.

If $\phi : T \times Y \to Y$ is a mapping , then for all $t \in T$ and $y \in Y$; we have $\phi(t, y)=t \cdot y$

Definition 1.1.2 (Cascade)**.** A *cascade* refers to a semiflow where a discrete topological action is applied on Y. This is applied mostly through a mapping $f : Y \to Y$ where for $n \in \mathbb{N}$; $n \cdot y=f^n(y)$ or f iterated n times. A cascade of this form is denoted by (Y, f)

Now, before defining chaos, we need to define its characteristics:

Definition 1.1.3 (Topological Transitivity)**.** A cascade (Y, f) is said to be *topologically transitive* if for any open,nonempty subsets $U, V \subset Y$, there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$

Thus, transitivity implies that for any two subsets, we can find a finite number of iterations so that the sets intersect.

Definition 1.1.4 (Devaney's Chaos)**.** we let $f : X \to X$ be a continuous transformation of a metric space X .

Devaney called it to be chaotic if it satisfies the following three conditions:

(i)$< X, f >$ is topologically transitive.

(ii) The set of periodic points of $< X, f >$ are are dense in X.

(iii) f is sensitive . [14]

We talk more about Devaney chaos in the following sections.

Example 1.1.1. *A tent map*, where $X = [0, 1]$ with standard topology and usual metric and $f(x) = min\{2x, 2(1 - x)\}$, *is topologically transitive.*

Example 1.1.2. *The logistic map*, where $X = [0, 1]$ with standard topology and usual metric and $f(x) = \mu x((1 - x)$ where $\mu \in (0, 4)$, *is not topologically transitive.*

Periodic Points

Definition 1.1.5 (Periodic Points)**.** Let (Y, f) be a cascade. A point $y \in Y$ is said to be periodic if $\exists n \in \mathbb{N}$ $s.t.$ $n \cdot x = x$ The set of periodic points of Y is denoted by $Per(Y)$

Example 1.1.3. In a tent map $x = 2/3$ is a periodic point.

Definition 1.1.6 (Preperiodic point)**.** A point x in a continuous self-map f is preperiodic if $\exists$ $m, n \in \mathbb{N}$ such $f^m(x) = f^{m+n}(x)$.

The orbit of a preperiodic point contains a periodic point.

Example 1.1.4. $X = [0, 1]$ and $f(x) = 2\min\{x, 1 - x\}$. Then orbit of 1/3 has $f(1/3) = 2/3$, where 2/3 is a fixed point.

Definition 1.1.7 (Recurrent point)**.** [2] A point $x \in X$ under f is recurrent if for any neighbourhood N of x and any $m \in \mathbb{N}$, $\exists n > m$ such that $f^n(x) \in N$.

A point is recurrent if it is an accumulation point of its orbit. Thus all periodic points are recurrent points.

Definition 1.1.8 (Almost periodic point)**.** [2] A point $x \in X$ under f is almost periodic if for any neighbourhood N of x $\exists m \in \mathbb{N}$ such that
$\{f^{n+i} : i \leqslant m\} \cap N \neq \phi, \ \forall n \in \mathbb{N}$.

Implications of transitivity

Sometime after R. L. Devaney incorporated the idea of a dense set of periodic points in the definition of chaos, [25] proved the redundancy of sensitivity.

If continuous $f : X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions.
[45] made a surprising discovery for continuous maps on intervals.

Let I be a (not necessarily finite) interval and let $f : I \to I$ be a continuous and topologically transitive map. Then periodic points of f are dense in I and f is sensitive as well. There was a general misunderstanding of the relation between transitivity and the existence of a dense orbit. [3] stated conditions under which the two are equivalent.

Let a complete metric space X with a countable base such that there is no dense subset of a non-empty open subset U and the function $f : X \to X$ is continuous. Then f is topologically transitive if and only if it has a dense orbit. The relationship between the dynamic of individual movement and the dynamic of collective movement was discussed by [40]. In essence, collective chaos implies individual chaos but not conversely.

If $f : X \to X$ is a continuous function, then if $\bar{f} : K(X) \to K(X)$ is transitive then f is also transitive, where $K(X)$ is the class of all non-empty and compact subsets of X and $\bar{f}$ is the natural extension of f to $K(x)$.

Stronger Forms of Transitivity and Invariance [5]

Definition 1.1.9. If A is subset of X then A is called *+invariant* if $f(A) \subset A$, *−invariant* if $f^{-1}(A) \subset A$ and *invariant* if $f(A) = A$.

Definition 1.1.10. Let X be a metric space and U and V be a pair of open and non-empty subsets of X. The system (X, f) is *exact* if, for every U and V, there exists a $n \in \mathbb{N}$ such that $f^n(U) \cap f^n(V) \neq \varnothing$.

The system (X, f) is *fully exact* if for every U and V there exist a $n \in \mathbb{N}$ such that $(f^n(U) \cap f^n(V))^0 \neq \varnothing$.

Definition 1.1.11. (X, f) is called

1. *Topologically Transitive* if $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in X for any U in X.
2. *Strongly transitive* if for every open and non-empty set U which is a subset of X and $\bigcup_{n=1}^{\infty} f^n(U) = X$.
3. *Very Strongly Transitive* if for every open and non-empty subset U of X, there exists a $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} f^n(U) = X$.
4. *Minimal* if there is no proper, nonempty, closed invariant subset of X.

5. *Weak Mixing* if $X * X$ or $f * f$ is topologically transitive.

6. *Exact Transitive* if for every open and non-empty subset pair U and V of X, $\bigcup_{n=1}^{\infty} f^n(U) \cap f^n(V)$ is dense in X.

7. *Strongly Exact Transitive* if for every open and non-empty subset pair U and V of X, $\bigcup_{n=1}^{\infty} f^n(U) \cap f^n(V) = X$.

8. *Strongly Product Transitive* if for every natural number n the product system (X^n, f^n) is strongly transitive.

9. *Mixing or Topologically Mixing* if, for every pair of open and non-empty subset U and V of X, there exists a $k \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geqslant k$.

10. *Locally Eventually Onto* if for every open and non-empty subset U of X, there exists a $n \in \mathbb{N}$ such that $f^k(U) = X \ \forall \ k \geqslant n$.

Example 1.1.5. Let $X = \{x \in \mathbb{N} \cap \{0\} | x \leqslant m\} \subset \mathbb{R}$, where m is any natural number. We define f as $f(i) = i + 1, \ i < m$ and $f(m) = 0$. Then f is locally eventually onto as $f^k(\{i\}) = X \ \forall \ k \ \geqslant m \ \forall i \in X$. Also, this means every orbit is dense, and hence f is minimal. Thus f is transitive.

Example 1.1.6. [38] Tent map $f = \min\{x, 1 - x\}$ on $X = [0, 1]$ is locally eventually onto. Let U be any non-empty open set. Let D be the set of all rationals in X with an odd numerator and denominator a power of 2 in reduced form. Then D is dense. Let $x = \frac{p}{2^m} \in D \cap U$, where p is odd and $m \geqslant 2$. Then $f^{m-2}(x)$ is $1/4$ or $3/4$, we let it be the former. Then any open interval containing $1/4$, by continuity will eventually map the whole space.

For open sets U and V, $\exists k, l$ such that k^{th} and l^{th} iterates of U and V are X respectively. Then f is strongly exact transitive as $f^r(U) \cap f^r(V) = X$, where

$r = \max(k, l)$.

Example 1.1.7. Let $x_0 \in X$ be a periodic point of f, and Y be the orbit of x_0. Let g be the restriction of f with respect to Y. Then g is minimal. Every point maps to every other point before coming back to itself after the number of iterates equal to the prime period of x_0.

Example 1.1.8. Let $X = [0, 1)$ and $f(x) = x + \alpha \bmod 1$, where α is an irrational in X. Then f is minimal. For this, we first show that 0 has a dense orbit. The orbit of 0 is infinite.

Let X be represented by the union of disjoint (except possibly at endpoints) intervals of equal width $1/M$. Let $f^m(0)$ and $f^n(0), n < m$, be in one of such intervals. Then the iterate g^{m-n} sends any point an arbitrarily small distance away, so the orbit of zero is dense. Now for any $x \in X$, $g^m(x) = g^m(0) + x,\ \forall m \in \mathbb{N}$, so x has dense orbit.

1.2 Sensitivity and its Stronger Forms

Chaos Theory is a Mathematical Theory that helps in the study of deterministic non-linear dynamical system [6] .

It is observed that in Chaos Theory, various dynamical systems can be differentiated based on their sensitivity. Sensitivity refers to the dependency of a dynamical system on its initial conditions. These initial conditions are varied. We can compare these variations and consequently differentiate the dynamical systems based on their sensitivity.

The most popular and mathematically rigorous definition of *sensitivty* was given by the American Mathematician John Guckenheimer in the late 1970s [13]. Robert L. Devaney used this definition to emphasize the importance of sensitivity in a chaotic system in Devaney's Chaos. [14]. The study of Sensitivity as a distinct area of research did not take place until the 2000s.Akin and Kolyada discovered Li-Yorke Sensitivity in 2003 [15]. Chinese Mathematician Xiong Jincheng discovered n-sensitivity in 2005 [15]. Indian Mathematician T K Subrahmonian Moothathu codified existing definitions of sensitivity and proposed stronger forms of sensitivity including multi-sensitivity in 2007 [17]

Definitions

Note: For easier understanding to the reader examples have also been given with each definition.

Definition 1.2.2. For A $\subset \mathbb{N}$, we say that A is cofinite if $\mathbb{N} \setminus$ A is finite.

Definition 1.2.3. A is thick if A contains arbitrarily large blocks of consecutive numbers.

Definition 1.2.4. : A is syndetic if N$\setminus$ A is not thick

Definition 1.2.5. A is piece-wise syndetic if it is an intersection of a syndetic set with a thick set.

Definition 1.2.6 (P-system)**.** It refers to the dynamical system, which is topologically transitive as well as has a dense set of periodic points.

Definition 1.2.7. When given TDS (A, g) we can say that, the given TDS is:

1. syndetically sensitive if $N(Y, \epsilon)$ is syndetic for some $\epsilon > 0$ and every nonempty open set $Y \subset A$.

2. thickly sensitive if $N(Y, \epsilon)$ is thick for some and $\epsilon > 0$ for every nonempty open set $Y \subset A$.

3. thickly syndetically sensitive if $N(Y, \epsilon)$ is thickly syndetic for some $\epsilon > 0$ and every nonempty open set $Y \subset A$.

4. thickly periodically sensitive if $N(Y, \epsilon)$ is thickly periodic for some $\epsilon > 0$ and every nonempty open set $Y \subset A$.

5. cofinitely sensitive if $N(Y, \epsilon)$ is cofinite for some $\epsilon > 0$ and every nonempty open set $Y \subset A$.

1.3 Chaos in Dynamical Systems

Mathematics studies several sets. Some of these sets, often known as systems, change with time. Such systems are known as dynamical systems. As given in [36], we deal with systems whose physical properties change with time t. We often specify the state at time t using variables:

$$y_1, y_2, \cdots, y_n$$

and their derivatives:

$$\frac{dy_1}{dt}, \frac{dy_2}{dt}, \cdots, \frac{dy_n}{dt}$$

As given in [37], dynamical systems have been in popular literature since the 19th century. Essentially, dynamical systems study the evolution of systems over time. A particular state of a dynamical system is referred to as the *orbit*. Dynamical systems are of 2 types: discrete and continuous.

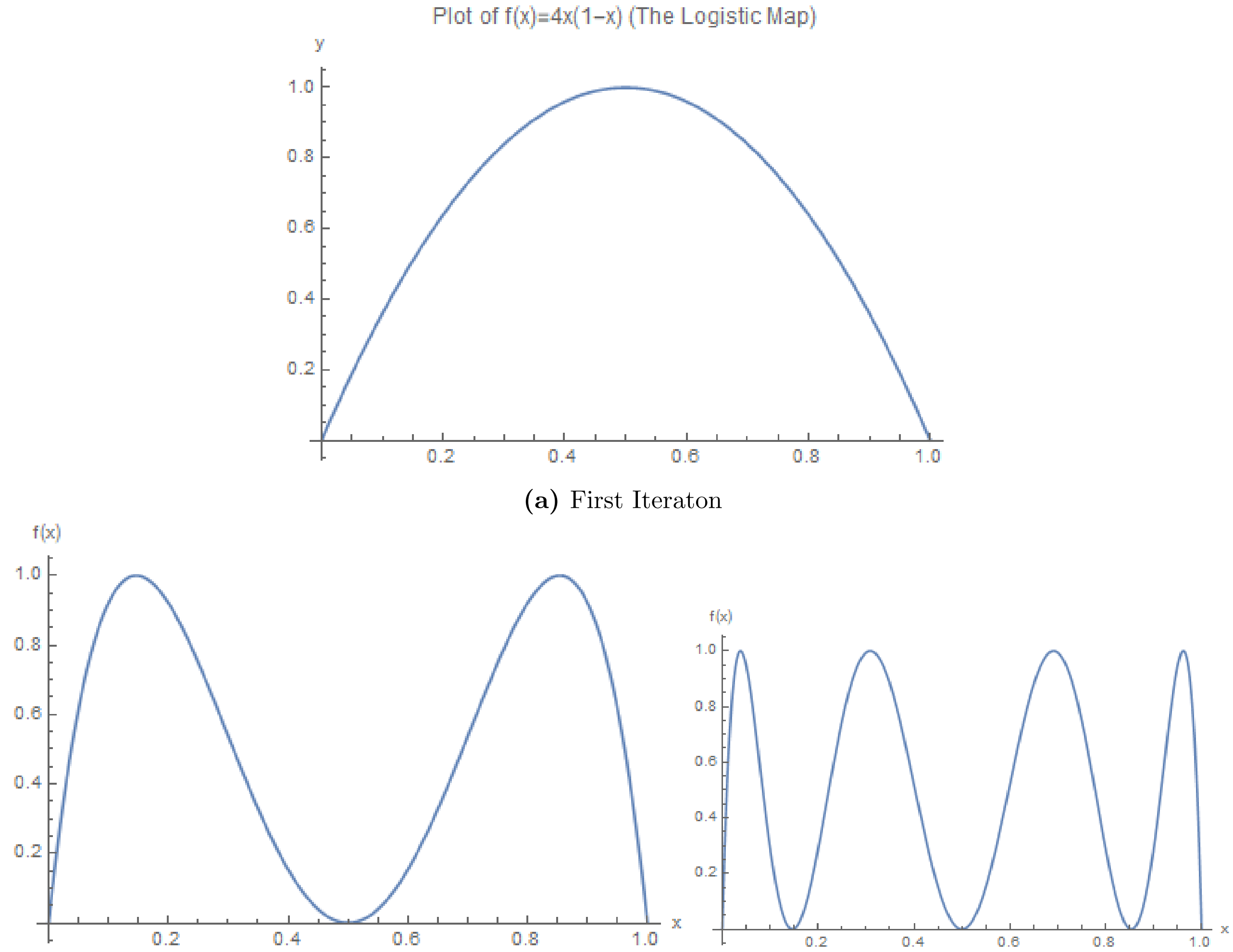
A *discrete* dynamical system consists of a non-empty set X and a mapping $f : X \to X$. We iterate this a finite number of times, $n \in \mathbb{N}$ because it is discrete. This system is also called *cascade*. Here f^0 denotes the *identity map*.

A *continuous-time* dynamical system, unlike a cascade, does not change with time intervals, it consists of a space X and a one-parameter family of maps $f^t : X \to X$ for $t \in \mathbb{R}$ that forms a one-parameter group or semi-group. This is called a *flow* if

$t \in \mathbb{R}$ and a semiflow if $t \in \mathbb{R}_0^+$.

A dynamical system is said to be *deterministic* when the system described by it follows a specific map completely determinable by its present state. An important feature of dynamical systems is *chaos.* This refers to the property of dynamical systems to exhibit dramatically different long-term behaviors due to a small change in initial conditions.

An important characteristic of long-term behaviors of dynamical systems is 'chaos'. According to [50], Derived from a Greek word , meaning 'a state without order'. This word relates to dismantled or unordered systems, contrary to the word *cosmos*, an ordered state. A more formal definition of chaos is 'an irregular oscillation or variation governed by a relatively simple rule'. Consider a simple map $f : [0, 1] \to \mathbb{R}$ given by $f(x) = 4x(1 - x)$ (the logistic map). Even with such a simple rule, as we will see further, the map appears to be irregular and uneven and hence, chaotic. If we take a particular $x_n \in [0, 1]$, and with successive iterations, we can observe chaotic or irregular behavior. For instance, consider the following Mathematica maps:



(a) First Iteraton

(b) Second Iteraton

(c) Third Iteraton

Figure 1.1: The Logistic Map

The variation is increasing with each iteration, this is precisely, what we mean by chaos. Several definitions have arisen in the study of chaos. Devaney [40], gave one of the most prominent definitions of chaos. He based it on 3 conditions, firstly, *topological transitivity* or the ability of 2 sets to coincide with time and iterations, secondly, *sensitivity* or the extent of increase in distance between 2 elements on successive iterations, and *Periodic points' denseness* which depicted the coverage of periodic points and their orbits in each subset. We talk more about this in further sections. There are other definitions of chaos which we examine in the below sections.

Preliminaries

To study chaos,we need to study some terms first. The definitions have been inspired from [38]

Definition 1.3.1 (Pointwise Sensitive)**.** Let $\mathcal{N}(y)$ denote the set of all neighborhoods of $y \in Y$. Let Y be a metric space with metric d. A cascade (Y, f) is called *pointwise sensitive* if $\exists c > 0$, such that $\forall y \in Y$, $\forall M \in \mathcal{N}(x), \exists n \in \mathbb{N}, x \in M$ satisfying $d(f^n(y), f^n(x)) \geqslant c$ So, pointwise sensitivity says that for any point in the metric space, we can find a point in the neighborhood whose distance keeps on increasing more than a sensitivity level (c) after n iterations.

Definition 1.3.2 (Setwise Sensitivity)**.** A metric space Y is said to be *setwise sensitive* if $\exists c > 0$ such that for any open nonempty $U \subseteq Y$, there exist $x, y \in U$ and $n \in \mathbb{N}$ satisfying $d(f^n(x), f^n(y)) \geqslant c$ According to this criterion, a matrix space Y is said to be setwise sensitive if, for any 2 points in it, the distance increases for a defined sensitivity level. So, both pointwise and setwise sensitive are equivalent.

Definition 1.3.3 (Dense Set)**.** Let Y be a metric space and $A \subseteq Y$, then the set

$$\overline{A} = \bigcap \{M | M \text{ is closed in } Y \text{ and } A \subset M\}$$

is called the closure of A. A subset A is called dense in Y if

$$\overline{A} = Y$$

We can also say that for any $x \in X$, and any $\epsilon > 0$, there is a point $a \in A$ such that $d(a, x) < \epsilon$.

Definition 1.3.4 (Lyapunov Exponent [43])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable map. Then $\forall x \in \mathbb{R}$, we define the local *Lyapunov exponent* of x, say, $\lambda(x)$ as:

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|$$

Definition 1.3.5 (Compact Metric Space [47])**.** A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact and considered as a subspace of X, that is, every subsequence in M has a convergent subsequence whose limit is an element of M.

Definition 1.3.6 (Isolated Point [38])**.** Let X be a topological space. A point $x \in X$ is called *isolated* if $\{x\}$ is an open set.

Definition 1.3.7 (Topologically Weakly Mixing [48])**.** Let X be a metric space and f be a map, then product map $f : X \to X$ is said to be topologically weakly mixing if $f \times f$ is topologically transitive.

Definition 1.3.8 (Topologically Mixing [48])**.** Let X be a metric space and $f : X \to X$ be a map. Now, define $N_f(U, V) = \{n \in Z^+ : f^n(U) \cap V \neq \emptyset\}$ Now, f is said to be topologically mixing if for any two nonempty open sets $U, V \subset X, N_f(U, V)$ is a *syndetic set.* In other words,there is an integer $M > 0$, such that $N_f(U, V) \cap n, n + 1, \ldots, n + M \neq \emptyset$

Definition 1.3.9 (Touhey Property [48])**.** Given a metric space X and a not-necessarily continuous semi-flow $\psi : R^+ \times X \to X$, we say that ψ has the Touhey property on X if, given U and V, nonempty subsets of X, there exists a periodic point $x \in U$ and a real number $t \geqslant 0$ such that $\psi(t, x) \in V$.

Definition 1.3.10 (Sequence Space)**.** The *sequence space* on two symbols is the set

$$\Sigma = \{(a_0 a_1 a_2 a_3 \ldots) | a_k = 0 \text{ or } 1\}$$

Thus, the space Σ is the set of all infinite sequences that can be made with 0 or 1. These are called *words*, or infinite words in 2 letters, 0 and 1.

The *distance* function between 2 elements $a = (a0 a_1 a_2 \ldots)$ and $b = (b_0 b_1 b_2 \ldots)$ of Σ is defined as:

$$d[a, b] = \sum_{k \geqslant 0} \frac{a_k - b_k}{2^k}$$

2 important results on sequence space are:

1. If $a_k = b_k$ for $k = 0, 1, \ldots, n$ then $d[a, b] \leqslant 1/2^n$

2. If $d[a, b] < 1/2^n$ then $s_k = t_k$ for $k = 0, 1, \ldots, n$.

The 2 properties are also called the *Proximity Theorem.*

Definition 1.3.11 (Shift map)**.** The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by:

$$\sigma(a_0a_1a_2\ldots) = (a_1a_2a_2\ldots)$$

Thus, the k^{th} *iteration* is given by:

$$\sigma^k(a_0a_1a_2\ldots) = (a_ka_{k+1}a_{k+2}\ldots)$$

Thus, the shift map denotes a dynamical system on the sequence space.

Observe that the periodic points of this system with period n are of the form

$$(a_0a_2a_2\ldots a_{n-1}a_0a_1\ldots a_{n-1}\ldots) = (\overline{a_0a_1\ldots a_{n-1}})$$

given that these can be 0 or 1, there can be 2^n points with period n and some of these will have prime period less than n. Thus, we can find eventually fixed and eventually periodic points in this system.

Definition 1.3.12 (Totally Transitive [49])**.** A map f on a metric space X is said to be totally transitive if f^n is transitive for all $n \in \mathbb{N}$. $x \in X$ is called a *fixed point* of f if $f(x) = x$.

Definition 1.3.13 (Ball [47])**.** Given a point x_0 (center) in a metric X, and a real number $r > 0$ (radius), we can define 3 types of sets:

- Open Ball: $B(x_0; r) = \{x \in X | d(x, x_0) < r\}$

- Closed Ball: $\overline{B}(x_0; r) = \{x \in X | d(x, x_0) \leqslant r\}$

- Sphere: $S(x_0; r) = \{x \in X | d(x, x_0) = r\}$

Definition 1.3.14 (G_δ Subset)**.** A subset of a topological space is said to be a G_δ subset if it is a countable intersection of open sets.

Definition 1.3.15 (Different relations between points [49])**.** Let (X, f) be a dynamical system with metric d. For 2 points x and y in X, they are called *asymptotic (AR)* if $\lim d(f^n(x), f^n(y)) = 0$. x and y are called *proximal (PR)* if $\liminf d(f^n(x), f^n(y)) = 0$. If they are not proximal, then the relation is called *distal (DR)*. For any relation R on $X \times X$, denote $R(x) = \{y : (x, y) \in R\}$ for any $x \in X$. Let $\Delta(n) = \{(x, y) : d(x, y) < 1/n\}$ and $\Delta = \{(x, x) : x \in X\}$. Let $A_{k,n} = \cap_{i=k}^{\infty}(f \times f)^{-i}\overline{\Delta_n}$. Note that each $A_{k,n}$ is closed, PR is a G_δ subset and AR is an equivalence relation on X.

Definition 1.3.16 (ω-Limit Sets [51])**.** The *ω-Limit Set* of a cascade f for a point $x \in X$ is given by :

$$\omega(x, f) = \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k \geqslant n\}}$$

Definition 1.3.17 (Topologically Conjugate [38])**.** Two function $f : A \to B$ and $g : C \to D$ are said to be *topologically conjugate* if there is a homomorphism $h : B \to C$ such that $h[f(a)] = g[h(b))]$ for all $a \in A, b \in B$.

An important property of topological conjugates is that they preserve topological properties like transitivity and denseness of periodic points. We will prove this result for transitivity, provided h is surjective.

Consider f, g be 2 maps on X and h be a surjective homomorphism as defined above. Now, let $A, B \subset X$ be arbitrary nonempty open subsets. Since h is surjective, $h^{-1}(A)$ and $h^1(B)$ are nonempty. And since f is topologically transitive, there exists an $n \in \mathbb{N}$ such that $f^n(h^{-1}(A) \cap h^{-1}(B)) \neq \emptyset$. Let $x \in h^{-1}(A)$ such that $f^n(x) \in h^{-1}(B)$. Hence, there is some $y \in A$ such that $h(x) = y$ and $g^n(y) = g^n(h(x)) = h(f^n(x))$. Since $f^n(x) \in h^{-1}(B)$, it follows that $g^n(y)h(f^n(x)) \in B$. Thus, it implies that g is also transitive.

Definition 1.3.18 (Unstable Points)**.** Let X be a compact metric space with metric d. Let $f : X \to X$ be a continuous map.A point $x \in X$ is said to *unstable* if there exists an $r > 0$ such that for every $\epsilon > 0$ there are $y \in X$ and $n \geqslant 1$ satisfying $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > r$.

Chapter 2

Transitivity and Density of Periodic Points

To understand the depth of concepts in a better way, it is crucial to comprehensively go through the crispness of topology, metric spaces, transitivity, periodic points, preperiodic points, recurrent points, invariance, etc in order to step up to what are known as stronger forms of transitivity. Various results allied to topological transitivity and density of periodic points include implications of transitivity (viz, sensitivity is redundant on infinite sets, transitivity implies chaos on interval, existence on dense orbits if and only if conditions of transitivity), transitivity in set-valued discrete system, lemmas and theorems on stronger forms of transitivity, period 3 or higher implies chaos. (After demonstrating that chaos implies a set of points being dense.)

2.1 Conditions for Transitivity and Existence of Dense Orbit

Topological transitivity and existence of dense orbit are two distinct concepts, and it is worth exploring when one implies the other. In general, they are not equivalent, but there are exceptions. We show that on a complete metric space with a countable base, the two concepts are equivalent.

The dynamics of the system can be studied by observing the trajectories of individual points in the system, which we can then use to infer the collective behavior of the system. However, this approach is limited in that it does not take into account the relationship between the individual elements. When we want to see the movement of sets rather than just of points, we study set-valued discrete systems. We are concerned about the relationship between the dynamics of individual movement and the dynamics of collective movement. We here see that transitivity of natural extension of f to class of all non-empty and compact subsets of X implies transitivity of f. In essence, collective chaos implies individual chaos but not conversely.

We first show some examples where neither Transitivity nor Existence of Dense Orbit implies the other.([3] [1])

Example 2.1.1. *[57]* Let $X = \{a, b\}$ with discrete topology and let $f : X \to X$ be the constant map to a. Then the orbit of b is dense in X. But $f^k(\{a\}) \cap \{b\}$ is nonempty for no k.

Example 2.1.2. On $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $f : X \to X$ be defined by $f(0) = 0$ and $f(1/n) = 1/(n + 1)$. Then 1 has a dense orbit. But f is not topologically transitive because for example, $U = \{1/2\}$ and $V = \{1\}$, then $(\bigcup_{n=1}^{\infty} U) \cap V = \phi$.

Example 2.1.3. Let $I = [0, 1]$ and $g : I \to I$ be a continuous map defined by $g(x) = 1 - |2x - 1|$. Let X be the set of periodic points of g. Let f be restriction of g on X, i.e., $f = g|_X$. Then f has no dense orbit since X is infinite as it is dense in I but any periodic point has a finite orbit.

However, it is topologically transitive. For any non-degenerate sub-interval J of I, there is a positive k such that $f^k(J) = I$. Hence, whenever J_1 and J_2 are nonempty open sub-intervals of I, there is a periodic orbit of g which intersects both J_1 and J_2.

Proposition 1. *Let X be the topological space that lacks a dense finite subset in any* *non-empty open subset U. A (not necessarily continuous) function $f : X \to X$ is topologically transitive if it meets this criterion and has a dense orbit.*

Proof. Let x have a dense orbit in X and let open, non-empty set $U, V \subset X$. Now we know that $f^m(x) \in U$ and $f^n(x) \in V$ (as x is dense in X).
First, we assume that, $m < n$ and let $k = n - m$. So, it is obvious that $f^k(U) \cap V \neq \varnothing$.

Now let us assume that $m \geqslant n$. As $m > n$, the orbit of x will enter set V many times before entering set U. Let's say these points are

$$f^{p_1}(x), f^{p_2}(x), ..., f^{p_k}(x)$$

$(n \leqslant p_i \leqslant m, i = 1, 2, ..., k)$. As these points are not dense in V there exists an open, non-empty subset V' of V such that none of these points exist in V'. As x is dense in X and $V' \subset V \subset X$, there exists a $l > 0$ such that $f^l(x) \cap V' \neq \varnothing$. l is greater than m, so for some $q = l - m$ we get $f^q(U) \cap V \neq \varnothing$
Hence, X is topologically transitive □

Proposition 2. *Let a complete metric space X with a countable base and a continuous function $f : X \to X$. Then f possesses a dense orbit if and only if f is topologically transitive.*

Proof. Let $(V_i)i$ be a countable base for X (here, $i \in I$). $W_i = \bigcup_{n \geqslant 0} f^{-n}(V_i)$ is open by continuity of f for $i \in I$. W_i is also dense in X.
Now to check this let there be a set U which is a non-empty open set. As f is topologically transitive there exits a $k > 0$ such that $f^k(U) \cap V \neq \varnothing$. This implies that $f^{-k} \cap U \neq \varnothing$ and $W_i \cap U \neq \varnothing$ Thus W_i is dense.
Let $B = \bigcap_{i \in I} W_i$ be dense in X by Baire category theorem. Now the orbit of any $x \in B$ is also dense in X. Because, for any subset U in X, there exists $i \in I$ such that

$V_i \subset U$ and $k > 0$ such that $x \in f^{-k}(V_i)$. This implies that $f^k(x) \in V_i \subset U$. Thus the orbit of x enters some arbitrary U. $\square$

Now, combining the above propositions

Proposition 3. *Let a complete metric space X with a countable base such that there is no dense subset of a non-empty open subset U and the function $f : X \to X$ is continuous. Then f is topologically transitive if and only if it has a dense orbit.*

Transitivity in Set-Valued Discrete System [4]

Lemma 1. *Let A be a non-empty open subset of X. Let $U \in K(X)$, where $K(X)$ is the class of all non-empty and compact subsets of X, and $U \subset A$ then there exist $\epsilon > 0$ such that, $\epsilon-$dilatation of U as a set, $N(U, \epsilon) = \{x \in X/(d(x, U) < \epsilon\} \subset A$.*

Proof. Let $b(A)$ be the boundary of A.We can assume that $b(A)$ is non-empty (since, $b(A) = \varnothing \implies A = X$ and here the result is obvious).
So, the map $h : U \to \mathbb{R}$, where $h(u) = d(u, b(A))$, is a continuous function. h will assume its minimum value at U.
Let $u_0 \in U$ such that $\delta = h(u_0) = min_{u \in U} h(u)$, Assuming $\delta = 0$ we get $u_0 \in b(A)$ (since A is open and b(A) is close) which is a contradiction and hence $\delta > 0$. Taking $0 < \epsilon < \delta$ we get $N(U, \epsilon) \subset A$. $\square$

Lemma 2. *Let A be a non-empty open subset of X. Then the extension of A,$e(A) = \{U \in K(X)/U \subset A$, is a non-empty open subset of $K(X)$.*

Proof. If $U \in e(A)$ then $U \subset A$. From 1 we know that there exists $\epsilon > 0$ such that $N(U, \epsilon) \subset A$. If $F \in B(U, \epsilon)$ and $H(U, F) = max\{\rho(U, F), \rho(F, U)\}$, where $\rho(U, F) = inf\{\epsilon > 0/U \subseteq N(F, \epsilon)\}$ and $\rho(F, U) = inf\{\epsilon > 0/F \subseteq N(U, \epsilon)\}$ then $H(U, F) < \epsilon \implies F \subset N(U, \epsilon) \subset A$ which further implies $F \in e(A)$. $\square$

Lemma 3. *If A is subset of X, then*

1. $e(A \cap B) = e(A) \cap e(B)$

2. $\bar{f}(e(A)) \subseteq e(f(A))$

3. $\bar{f}^p = \bar{f}^p$ *for every* $p \in \mathbb{N}$.

Proof. 1. $U \in e(A \cap B) \iff U \subset A \cap B \iff U \in e(A) \cap e(B)$

2. If $U \in \bar{f}(e(A))$ then there exists a $U_1 \in e(A)$ such that $U = \bar{f}(U_1) = \{f(x)/x \in U_1\}$. Since, $U_1 \subset A$, we get $U \subset f(A) \implies U \in e(f(A)$.

3. Obvious.

□

Theorem 2.1.1. If $f : X \to X$ is a continuous function, then if $\bar{f} : K(X) \to K(X)$ is transitive then f is also transitive.

Proof. Let P and Q be two non-empty open subsets of X. By 2, we know that $e(P)$ and $e(Q)$ are non-empty open sets in $K(X)$. So, by transitivity if $\bar{f}$, we have $\bar{f}^n(e(P)) \cap e(Q) = \bar{f}^n(e(P)) \cap e(Q) \neq \varnothing$, for some $n \in \mathbb{N}$.
By 3, we can say, $e(f^n(P)) \cap e(Q) = e(f^n(P) \cap Q \neq \varnothing$. Since $e(A) = \varnothing \iff A = \varnothing$, we get $f^n(P) \cap (Q) \neq \varnothing$ which implies f is transitive. □

2.2 Stronger Forms of Transitivity

Many different forms and definitions of stronger form of transitivity are prevalent in literature. Here we strive to give unifying definitions for these concepts while studying their relations. For instance, on an open map, very strongly transitivity is equivalent to strongly transitivity while both imply transitivity. Similarly, if the space is minimal, then it is very strongly transitive. On a closed set with no isolated points, these different forms coincide.

Lemma 4. *If (A_n) is a sequence of subsets of X then*

(a) For every ϵ there exist $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} A_n$ is ϵ dense if and only if $\bigcup_{n=1}^{\infty} A_n$ is dense.

(b) For all open A_n and $\bigcup_{n=1}^{\infty} A_n = X$ there exist a $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} A_n = X$.

Proof. (a) Because X is compact it has a finite cover by $\epsilon/2$ balls. A set that meets each of these is ϵ dense.

(b) This follows from compactness. □

Theorem 2.2.1. (a) If (X, f) is an exact system and f is one-to-one then X is a singleton, i.e. the system is trivial.

(b) (X, f) is fully exact if and only if for every pair of U and V which are open and non-empty subsets of X, $\bigcup_n (f^n(U) \cap f^n(V))^0 \neq \varnothing$

Proof. (a) Let X is not a singleton which means it has a pair of open and non-empty sets U and V which are disjoint. Since, (X, f) is exact we have $f^n(U) \cap f^n(V) \neq \varnothing$ for some $n \in \mathbb{N}$ which implies that f is not one-to-one which is a contradiction. Hence, X is a singleton.

(b) Since (X, f) is fully exact we have, $\bigcup_n (f^n(U) \cap f^n(V))^0 \subset (\bigcup_n f^n(U) \cap f^n(V))^0$ which implies that the latter is non-empty.

Let $P \in U$ and $Q \in V$ be closed and non-empty sets. Let an open set $A = (\bigcup_n f^n(U) \cap f^n(V))^0$.Since it is non-empty it is Baire space with a countable, relatively closed over $\{A \cap f^n(P) \cap f^n(Q) : n \in \mathbb{N}\}$. So, $f^n(P) \cap f^n(Q)$ has a non-empty interior in A and so in X (by Baire Category Theorem). In fact, $\bigcup_n (f^n(P) \cap f^n(Q))^0$ is dense in A. It implies that (X, f) is fully exact. □

Theorem 2.2.2. *Let (X, f) be a dynamical system. The following are equivalent.*

1. The system is topologically transitive.

2. There exists a $n \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \varnothing$, for some U and V open and non-empty subset of X.

3. For every pair of open and non-empty subsets U and V of X the set $N(U,V)$ (where it is a time-hitting set, i.e. $N(U,V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \varnothing\} = \{n \in \mathbb{N} : U \cap f^{-n}(V) \neq \varnothing\}$) is non-empty.

4. For every pair of open and non-empty subsets U and V of X the set $N(U,V)$ is infinite.

5. For some $x \in X$ the orbit, $O(x)$ is dense in X, i.e. the set of transitive points, $Trans(f)$, is non-empty.

6. $Trans(f) = \{x : w(x) = X\}$ is dense G_δ subset of X.

7. $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in X for any U in X.

8. For every open and non-empty subset U of X and $\epsilon > 0$ there exist a $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} f^n(U)$ is ϵ dense in X.

9. $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X for any U in X.

10. For every open and non-empty subset U of X and $\epsilon > 0$ there exist a $k \in \mathbb{N}$ such that $\bigcup_{n=1}^{k} f^{-n}(U)$ is ϵ dense in X.

11. Let U be an open and non-empty subset of X and if U is -invariant then U is dense in X.

12. If A is a closed subset of X and +invariant, then either $A = X$ or A is nowhere dense in X.

If the system (X, f) is topologically transitive, then f is surjective, i.e. onto, and X is either a single periodic orbit or perfect space, i.e. has no isolated points.

Proof. We can see that condition (7) is the definition of Topological Transitivity, so (1) $\iff$ (7).

(7) $\implies$ $f(X)$ is dense and by compactness equals X, i.e. f if surjective.

Now, let us assume that for $x \in X$ orbit $O(x)$ is dense. Now, if $x \in O(x)$ then x is a periodic point with finite orbit $O(x)$ dense in X and equals $w(x)$ which means X is a periodic point. If $x \notin O(x)$ then it implies that $O(x)$ is dense but not closed and is infinite which means that all the points of the orbit are distinct. We get for some $y \in X/O(x)$, it is the limit point of some sequence $f^{n_i}(x)$ where $n_i \in \mathbb{N}$ and $n_i \to \infty$. In particular, there is such a sequence with $f^{n_i}(x) \to x$ and so $f^{n_i+k}(x) \to f^k(x)$ $\forall\ k \in \mathbb{N}$. Thus no point of X is isolated and every point is contained in $w(x)$. Each of $(2), (7), (9) \iff (3)$ is any easy exercise. Since $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is -invariant and equals U if U is invariant, and hence (9) $\iff$ (11). By Lemma 4 (a).

(7) $\iff$ (8) and (9) $\iff$ (10). (11) $\iff$ (12) by taking compliments.

(4) $\implies$ (5) and (4) $\implies$ (3) are obvious.

It's clear that $Trans(f) = \bigcap_U \bigcup_{n=1}^{\infty} f^{-n}(U)$, with you varying over countable base. By assumption (9) each $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is a dense open set. By Baire Category Theorem, $Trans(f)$ is a dense G_δ set. By initial argument,

$Trans(f) = \{x : w(x) = X\}$. This gives (9) $\implies$ (6).

(5) $\implies$ (4): (4) is obvious if X is a periodic orbit. Otherwise, (5) and our initial argument implies that X is perfect. If $O(x)$ is dense then it meets every open and non-empty set in an infinite set because X is perfect. It then follows that $N(U, V)$ is infinite for every open and non-empty pair of U and V. $\square$

Corollary 0.1. *For a system (X, f), the set of transitive points, i.e. $Trans(f)$ is invariant and -invariant.*

Every transitive point is recurrent.

Proof. By (6) in Theorem 2.2.2 $Trans(f) = \{x : w(x) = X\}$ and $w(x) = w(f(x))$ as $w(x)$ is invariant. It implies that $Trans(f)$ is invariant and -invariant. Now, $x \in w(x)$ implies every transitive point is recurrent. $\square$

Theorem 2.2.3. Let (X,f) be a dynamical system, then the following are equivalent.

1. The system (X,f) is strongly transitive.
2. For every open and non-empty subset U of X and for every $x \in X$, there exists a $n \in \mathbb{N}$ such that x belongs to $f^n(U)$.
3. For every open and non-empty subset U of X and for every $x \in X$, the set N(U,V) is non-empty.
4. For every open and non-empty subset U of X and for every $x \in X$, the set N(U,V) is infinite.
5. The negative orbit $O^-(x)$ is dense in X $\forall\, x \in X$.
6. For all $x \in X$ and $\epsilon > 0$ there exist a $n \in \mathbb{N}$ such that $O_n^-(x)$ is ϵ dense in X.
7. If A is a non-empty subset of X and -invariant, then A is dense in X.

(X,f) strongly transitive implies f is topologically transitive.

Proof. We can easily find that $(1), (3)\ and\ (5) \iff (2)$ and $(5) \iff (6)$ by Lemma 4.

$(4) \implies (3)$ obvious.

$(5) \implies (4)$: If $n \in N(U, x)$ then there exists a $y \in U$ with $f^n(y) = x$. Because $O^-(y)$ is dense there exists $k \in N(U, y)$. That is, there exists $z \in U$ such that $f^k(z) = y$ and so $f^{k+n}(z) = x$. Hence, $k + n \in N(U, x)$. Thus, the set $N(U, x)$ is unbounded.

$(5) \implies (7)$: $O^-(x)$ is -invariant and if $x \in A$ and A is -invariant then $O^-(x) \subset A$.

Condition (3) $\implies$ condition (3) of theorem 0.1 and hence Strongly Transitive implies Topological Transitive. $\square$

Theorem 2.2.4. *Let (X,f) be a dynamical system, and the following are equivalent for a dynamical system.*

1. The system is strongly transitive.
2. For every $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $O_k^-(x)$ is ϵ dense in X for every $x \in X$.

If (X,f) is very strongly transitive then for every open and non-empty subset U of X and every point $x \in X$ the set N(U,x) is syndetic.

Proof. (1) $\implies$ (2): Cover X by $\epsilon/2$ balls $V_1, ..., V_m$. There exist an $k \in \mathbb{N}$ large enough that $\bigcup_{n=1}^{k} f^n(V_i) = X$, for $i = 1, ..., m$. Fixing $x \in X$, we get a $y \in V_i$ for some i and some $y \in X$ and $x \in f^n(V_i)$ for some n such that $1 \leqslant n \leqslant k$. Therefore, $O_k^-(x)$ is ϵ dense in X.

(2) $\implies$ (1): For some $k \in \mathbb{N}$ there exists a $\epsilon > 0$ such that $O_k^-(x)$ is ϵ dense in X for every $x \in X$. Let W be ϵ dense in X. For $x \in X$ there exists a $x' \in W$ such that $f^n(x') = x$ where $1 \leqslant n \leqslant k$. Hence $x \in \bigcup_{n=1}^{n} f^n(W)$. Since x is arbitrary $\bigcup_{n=1}^{k} f^n(W) = X$.
If $X = \bigcup_{n=1}^{k} f^n(U)$ then for every $m \in \mathbb{N}$, $X = f^m(X) = \bigcup_{n=1}^{k+m} f(U)$. Thus for every $x \in X$ the set $N(U, x)$ meets every interval of length k in $\mathbb{N}$. $\square$

Corollary 0.2. *Let (X, f) be a dynamical system. If (X, f) is strongly transitive, then the set $N(U, V)$ is syndetic for any open and non-empty subset U and V of X.*

Proof. If $x \in V$, then $N(U, x) \subset N(U, V)$. $\square$

Theorem 2.2.5. For an open map f the following conditions are equivalent.

1. The system is very strongly transitive.

2. The system is strongly transitive.

3. X does not contain a proper, closed and -invariant subset.

Proof. (1) $\implies$ (2) $\implies$ (3) whether the map is open or not.

(2) $\implies$ (1): U, open and non-empty implies $f^n(U)$ is open and if $\{f^n(U) : n \in \mathbb{N}\}$ covers X then it has finite subcover.

(3) $\implies$ (2): If A is a non-empty and invariant subset of X then , $\bar{A}$ is a non-empty and closed -invariant subset of X and so it equals X. Thus, A is dense by Theorem 2.2.3 (6). □

Theorem 2.2.6. If (X, f) is a dynamical system, then the following are equivalent.

1. The system is weakly mixing.

2. For a triple of open and non-empty subsets $U, V, W \subset X$, there exists $n \in \mathbb{N}$ such that $f^{-n}(U) \cap W \neq \varnothing$ and $f^{-n} \cap W \neq \varnothing$.

3. For a triple of open and non-empty subsets $U, V, W \subset X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap W \neq \varnothing$ and $f^n \cap W \neq \varnothing$.

4. $\forall n \in N$ the product system (X^n, f^n) is topologically transitive.

5. For every open and non-empty set $U \subset X$, and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f^{-n}(U)$ is ϵ dense in X.

6. For every open and non-empty set $U \subset X$, and $\epsilon > 0$, f^{-n} is ϵ dense in X for infinitely many $n \in \mathbb{N}$.

7. For every open and non-empty set $U \subset X$, and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f^n(U)$ is ϵ dense in X.

8. For every open and non-empty set $U \subset X$, and $\epsilon > 0$, f^n is ϵ dense in X for infinitely many $n \in \mathbb{N}$.

Proof. (2) and (3) are the characterization of weakly mixing, so (1) $\iff$ (2)*and*(3).
(1) $\iff$ (4) is a consequence of Furstenberg Intersection Lemma.
(5) $\implies$ (2) *and* (7) $\implies$ (3) Choosing $\epsilon > 0$ small enough that ϵ ball is contained in both V and W.

(6) $\implies$ (5) *and* (8) $\implies$ (7) Obvious
(4) $\implies$ (6) *and* (8) Let $V_1, ..., V_m$ be finite cover of X by $\epsilon/2$ balls. Since the product system (X^m, f^m) is topologically transitive, there exists infinitely many n_1, n_2 such that $n_1 \in N(U, V_1) \cap \cdots \cap N(U, V_m)$ and $n_2 \in N(V_1, U) \cap ... \cap N(V_m, U)$. This implies that $f^{n_1}(U)$ and $f^{-n_2}(U)$ are ϵ dense. □

Theorem 2.2.7. Let a dynamical system (X, f)

(a) If the system (X, f) is exact transitive then it is weak mixing.

(b) The following are equivalent

(1) The system (X, f) is strongly exact transitive.

(2) For all pairs of open sets U and V, $\bigcup_{n \in \mathbb{N}} (f * f)^n (U * V)$ contains the diagonal id_X.

(3) For all $x \in X$, the negative $(f * f)$ orbit $O^-(x, x)$ is dense in $X * X$.

If (X,f) is strongly exactly transitive, then it is exactly transitive and strongly transitive.

Proof. (a) It clearly holds from condition (12) of Theorem 2.2.6.

(b) All three conditions imply that for every $x \in X$ and for every open and non-empty subset U and V of X, there exists $n \in \mathbb{N}$ such that $x \in f^n(U)$ and $x \in f^n(V)$. $\square$

Theorem 2.2.8. (a) If (X, f) is exact transitive and fully exact then it is exact transitive

(b) If the system (X, f) is topologically transitive and fully exact then the system is exact transitive.

(c) If (X, f) is strongly exact transitive then it is fully exact as well.

Proof. (a): Obvious

(b): Let the system (X, f) is topologically transitive and fully exact. For an open and non-empty pair of set U and V, there exists a transitive point x in the open and non-empty set $(\bigcup_n f^n(U) \cap f^n(V))^0$. So there exists a $n \in \mathbb{N}$ such that $x \in f^n(U) \cap f^n(V)$. The orbit $O(x)$ is then contained in $\bigcup_{k \geqslant 0} f^k(U) \cap f^k(V)$ and the latter set is dense.

(c) This follows from Theorem 2.2.1 (b). $\square$

Theorem 2.2.9. Let a dynamical system (X, f). The following are equivalent for the dynamical system.

1. The system (X, f) is topologically mixing.
2. For every open and non-empty pair of subset U and V of X, the set $N(U, V)$ is co-finite.
3. For every pair of open and non-empty subset U of X and $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $f^{-n}(U)$ is ϵ dense in X $\forall\ n \geqslant k$.

4. For every pair of open and non-empty subset U of X and $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $f^n(U)$ is ϵ dense in X $\forall$ $n \geqslant k$.

If the system $(X.f)$ is topologically mixing then it is weak mixing.

Proof. (1) $\iff$ (2) and (3), (4) $\implies$ (2) are obvious.
(2) $\implies$ (3) and (4): Let $V_1, ..., V_m$ be finite cover of X by $\epsilon/2$ balls. As $N(U, V_1), ..., N(U, V_m)$ and $N(V_1, U), ..., N(V_m, U)$ are co-finite, there intersection $N(U, V_1) \cap \cdots \cap N(U, V_m)$ and $N(V_1, U) \cap \cdots \cap N(V_m, U)$ are also finite. □

Theorem 2.2.10. If (X, f) is a dynamical system then the following are equivalent.

1. The system (X, f) is locally eventually onto.

2. For all $\epsilon > 0$, there exists a $n \in \mathbb{N}$ such that $f^{-n}(x)$ is ϵ dense in X for every $x \in X$.

3. For all $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $f^{-n}(x)$ is ϵ dense in X for every $n \geqslant k$.

If the system (X, f) is locally eventually onto then it is strongly product transitive and topologically mixing.

Proof. We know that if $f^k(U) = X$ then $f^n(U) = U$ $\forall$ $n \geqslant k$.
(1) $\implies$ (3): Let $\{U_1, ..., U_m\}$ be a cover by $\epsilon/2$ balls. There exists a $k \in \mathbb{N}$ such that for all $n \geqslant k$, $f^n(U_i) = X$ for every $i = 1, .., m$. Then $f^{-n}(x)$ meets each U_i $\forall$ $x \in X$ and $n \geqslant k$. So all such $f^{-n}(x)$ are ϵ dense.
(3) $\implies$ (2): Obvious
(2) $\implies$ (1) It is given that U is open and non-empty. Now let $\epsilon > 0$ such that U contains an ϵ ball. If $f^{-n}(x)$ is ϵ dense $\forall$ x then $f^{-k}(x)$ meets U for every x. Thus, $f^k(U) = X$. □

2.3 Period 3 and Higher Implies Chaos

If f is chaotic on a compact metric space with no isolated points, then the set of points with period less than n is nowhere dense, i.e., set of points with period at least n is dense for each n. Conversely, if f is a continuous function from a closed interval to itself, for which the set of points with period at least n is dense for each n, then there is a decomposition of the interval into closed subintervals on which either f or f^2 is chaotic. Finally, absence of proper invariant non-degenerate sub-interval on f and f^2 along with set of points with period at least 3 being dense, assure chaos on f. We first show that chaos implies that a set of points with a prime period at least n is dense. [54]

Theorem 2.3.1. Let X be a compact metric space without isolated points and f be a chaotic map. Then P_n, the set of points with prime period at least n is dense.

Proof. Assuming otherwise, let $U = \bar{P_n}^C$. Then all open sets in U have periodic points with periods strictly less than n. Let z have dense orbit and $z_1 = f^k(z) \in U$. We define $z_i = f^{k+i-1}(z),\ i \leqslant n$. Let ϵ be the minimum distance between any two z_i, then we define open $B_i = B(z_i, \epsilon/2)$ which are mutually disjoint. Let
$V_n = B_n;\ V_i = B_i \cap f^{-1}(V_{i+1}),\ i < n$. Then V_i contains z_i, is open and $f(V_i) \subseteq V_{i+1}$. Now, $z_1 \in V_1 \cap U$ so it is non-empty and contains a periodic point with a period strictly less than n, say y. But $y \in V_1$, so $f^i(y) \in V_{i+1}$, so y has a period at least n, which is a contradiction. □

The density of P_n doesn't imply chaos.

Example 2.3.1. Let D be the closed unit disk in the complex plane. Define $f : D \to D$ by $f(re^{i\theta}) = re^{i(\theta+2\pi r)}$. Then every rational r is periodic with a period at most the denominator of r. Then P_n is dense for all n as a set of rationals with denominator at least n is dense. $d(f^n(0), f^n(z)) = |z|\ \forall n$ so D is not sensitive to initial condition. Also D is not transitive as $f^n(B(1/2, 1/8)) \cap B(3/4, 1/8) = \phi\ \forall n$.

Lemma 5. *Let continuous $f : [0,1] \to [0,1]$ have a dense set of periodic points. Suppose that $0 \leqslant a < b \leqslant c < d \leqslant 1$ and that f permutes the intervals $[a,b]$ and $[c,d]$. Then $f(b) = c,\ f(c) = b$, and f fixes $[b,c]$. Moreover, $0 < a$ if and only if $d < 1$, in which case $f(a) = d,\ f(d) = a$, and f permutes the intervals $[0,a]$ and $[d,1]$.*

Proof. If $x \in ([a,b] \cup [c,d])^C$, and $f(x) \in [a,b] \cup [c,d]$ then x is not periodic as $[a,b] \cup [c,d]$ is invariant. If a point of (b,c) is mapped outside the interval, then as endpoints of this interval are mapped into $[a,b] \cup [c,d]$, and f is continuous, so some sub interval of $[b,c]$ would be mapped into $[a,b] \cup [c,d]$. This sub-interval then cannot contain a periodic point which is contrary to the hypothesis. Hence $[b,c]$ is invariant. Also, $f(b) \in [c,d] \cap [b,c] = c$. Hence, $f(b) = c$ and similarly $f(c) = b$.

□

Lemma 6. *Let continuous $f : [0,1] \to [0,1]$ have a dense set of periodic points. If f has no proper invariant non degenerate sub interval $[a,b]$, then either:*

1. $0 = a < b < 1,\ (b) = b$ and $[b,1]$ is also invariant; or

2. $0 < a < b = 1,\ f(a) = a$ and $[0,a]$ is also invariant; or

3. $0 < a < b < 1,\ f(a) = a,\ f(b) = b$ and the intervals $[0,a]$ and $[b,1]$ are also invariant under f ; or

4. $0 < a < b < 1,\ f(a) = b,\ f(b) = a$ and f permutes the intervals $[0,a]$ and $[b,1]$, which are also invariant underf^2.

Proof. Let $0 < a < b < 1$ and $a < f(a) < b$. Then by continuity, $(a - \epsilon, a)$ maps to (a,b) for some $\epsilon > 0$ which would then contain no periodic point, which is a contradiction. Hence a maps to a or b. A similar conclusion follows for b. Let $f(a) = a$. Then $[0,a]$ is invariant otherwise some of its sub-interval will map into $[a,b]$. If $f(b) = a$, then some sub interval of $[b,1]$ maps into $[b,1]^C$ which is invariant.

Hence $f(b) = b$ and $[b, 1]$ is invariant. If $f(a) = b$, then $[0, a]$ maps to $[b, 1]$ otherwise, again some of its sub-interval will map into $[a, b]$. Similarly, $[b, 1]$ can map either to $[b, 1]$ or $[0, a]$. Former is impossible as $[0, a]$ maps its periodic points to $[b, 1]$. □

Proof of other cases is similar.

Lemma 7. *Let continuous $f : [0, 1] \to [0, 1]$ have a dense set of periodic points. If f has no proper invariant non-degenerate sub-interval, then either:*

1. f is transitive; or

2. f permutes two sub intervals $[0, a]$ and $[a, 1]$ for some $0 < a < 1$.

Proof. Let f have no invariant sub-interval and it is not transitive. Let U, V be open intervals such that $f^n(U) \cap V = \phi\ \forall n$. Let $U' = \cup_{i \leqslant n} f^i(U)$. Then, $U' \cap V = \phi$. U' can be represented as a union of disjoint intervals.

Let J be a sub-interval of U'. Then $J = \cup_{i \in A} f^i(U)$, where A is some subset of natural numbers. $f(J) = \cup_{i \in A} f^{i+1}(U) \subseteq U'$. Also for some periodic point in J of order say, k, $f^k(J) \subseteq J$ since $f^k(J)$ is a sub interval of U'. Thus $\bar{U}'$ can be represented as finite union of closed intervals $I_i = [r_i, s_i],\ s_i \leqslant r_{i+1},\ 1 \leqslant i \leqslant n$ that are permuted by f. Also, $\bar{U}'$ cannot be an interval due to non-existence of invariant sub-intervals. Let k be the least number for which $f(I_k) = I_j,\ j < k$. Also let $f(I_{k-1}) = I_m$. We will show that I_k and I_{k-1} permute. Then $n = 2$ i.e., $[r_1, s_1]$ and $[r_2, s_2]$ are the only intervals. Hence $[0, s_1]$ and $[r_2, 1]$ permute by a previous lemma and $[s_1, r_2]$ is degenerate. For this, we consider three cases: (a) $j = k - 1$ and $k = m$; (b) $j \leqslant k - 1$ and $m > k$; (c) $j < k - 1$ and $m \geqslant k$. In the third case, $f(s_{k-1}) \geqslant r_k$ and $f(r_k) \leqslant r_{k-1}$. This means some sub interval of $[s_{k-1}, r_k]$ maps to I_{k-1}, contradicting density of periodic points. The second case is impossible similarly. □

Lemma 8. *If $P_n(f) = \{x : x$ has prime period at least n with respect to $f\}$, then*

$P_{nk}(f) \subseteq P_n(f^k)$. Also, if $P_3(f)$ is dense, then both f and f^2 have dense set of periodic points.

Proof. Let $x \in P_{nk}(f)$ with period m. Then $f^{(k)m}(x) = x \implies x \in P_n(f^k)$ as $m > n$. Thus $P_{nk}(f) \subseteq P_n(f^k)$.

$$\begin{aligned} P_3(f) &= P_4(f) \cup \{x : x \text{ has period 3 with respect to } f\} \\ &\subset P_2(f^2) \cup \{x : x \text{ has period 3 with respect to } f\} \\ &\subset P_2(f^2), \end{aligned}$$

because any element of period 3 in f has period 3 in f^2 as well. □

Theorem 2.3.2 (Period 3 and higher implies chaos)**.** Let P_3 be dense with respect to a continuous map $f : [0, 1] \to [0, 1]$. Then one of the following is true:

1. There is a finite or countably infinite collection $\mathcal{I}$ of non-degenerate closed intervals of $[0, 1]$ such that:

   (a) union of all intervals in $\mathcal{I}$, $\bigcup_{I \in \mathcal{I}} I$ is dense in $[0, 1]$;

   (b) intervals in $\mathcal{I}$ meet only at the endpoints;

   (c) each interval in $\mathcal{I}$ is invariant with fixed endpoints, except when endpoint are 0 or 1;

   (d) f is chaotic on each interval in $\mathcal{I}$.

2. There is a finite or countably infinite collection $\mathcal{I}$ of non-degenerate closed intervals of $[0, 1]$ and a central interval A, which might be degenerate such that:

   (a) f^2 and $\mathcal{J}$ satisfy all conditions in (1) above, where $\mathcal{J}$ is $\mathcal{I}$ if A is degenerate and $\mathcal{J}$=$\mathcal{I} \cup A$ otherwise;

   (b) f fixes A and acts as an order-reversing permutation of the intervals in $\mathcal{I}$, transposing them in pairs about A.

Proof. Since $P_3(f)$ is dense, both f and f^2 have a dense subset of periodic points that are not fixed, and therefore set of fixed points is nowhere dense.

Let f have a proper non-degenerate invariant sub-interval [a,b]. Then by one of the previous lemma, we have four cases:

- $[0,b]$ and $[b,1]$ are invariant and non-degenerate;
- $[0,a]$ and $[a,1]$ are invariant and non-degenerate;
- $[a,b]$, $[0,a]$ and $[b,1]$ are invariant and non-degenerate;
- $[a,b]$ is invariant and $[0,a]$ and $[b,1]$ permute, neither of which are degenerate.

In the first three cases, at least one of the points a and b is not equal to 0 or 1 and is fixed, f does not permute any sub intervals of $[0,1]$ in pairs. For example, in the third case, if $[c,d]$ and $[e,f]$ permute with $d < e$, $c \neq 0$ and $f \neq 1$, then $[0,c]$ and $[f,1]$ permute. This would mean that some sub interval $[0,a]$ maps outside $[0,a]$.

Let J be the set of endpoints of all proper non-degenerate invariant closed sub-intervals. Then all elements of J, except possibly 0 or 1 are fixed, and hence J is nowhere dense. Thus we can construct a collection of proper non-degenerate invariant closed sub-intervals $\mathcal{I}$ with the property that no element contains a proper invariant non-degenerate sub-interval. 1(a) follows, for example in the third case, $\bigcup_{I\in\mathcal{I}} I = [0,a] \cup [a,b] \cup [b,1]$. For 1(d), we see that transitivity and hence chaos follow on each sub-interval due to a lack of intervals permuting in pairs.

In the fourth case, let $\mathcal{K}$ be the collection of all closed invariant proper sub intervals $[g,h]$, not containing 0 or 1, such that $[0,g]$ and $[h,1]$ permute. Also, if $[g_1,h_1]$ and $[g_2,h_2]$ are in $\mathcal{K}$, then one is a subset of the other because of the density of periodic points. Let $A = [\alpha,\beta]$ be the intersection of all members of $\mathcal{K}$, which can be degenerate.

Then A is invariant and $[0,\alpha]$ and $[\beta,1]$ permute. Thus, A, $[0,\alpha]$ and $[\beta,1]$ are invariant under f^2 and a and b are fixed. This is similar to cases 1-3 above depending on whether α and β are equal or not, with f replaced by f^2. Thus we can find a collection of proper non-degenerate invariant closed sub-intervals $\mathcal{I}$ with respect to f^2, satisfying 1 of the theorem. $\mathcal{I}$ contains A and subsets of $[0,\alpha]$ and $[\beta,1]$. Also, under f, A is invariant and all other members of $\mathcal{I}$ permute. But since f permutes $[0,\alpha]$ and $[\beta,1]$, these intervals must be permuted in pairs satisfying 2(b).

Let f have a proper non-degenerate invariant sub-interval. If f is transitive, we are done. Otherwise $\exists a$ such that f permutes two sub-intervals $[0,a]$ and $[a,1]$ and we reach a similar situation as in the previous paragraph with $\alpha=\beta=a$. $\square$

Corollary 0.3. *Consider a continuous map $f:[0,1]\to[0,1]$ has a dense set of periodic points of period at least 3. If both f and f^2 have no proper invariant non-degenerate sub-interval, then f is chaotic.*

Example 2.3.2. A tent map function on $[0,1]$ with P_3 dense (tent map is chaotic) and 0 not a fixed point is

$$f(x)=\begin{cases}\frac{4}{3}x+\frac{2}{3}, & \text{if } x\in\left[0,\frac{1}{4}\right],\\ -\frac{4}{3}x+\frac{4}{3}, & \text{if } x\in\left(\frac{1}{4},1\right].\end{cases}$$

Conclusion

In this chapter we studied the interrelationship between transitivity and density of periodic points. By proving **Theorem 2.2.1**, **Theorem 2.2.2** and **Theorem 2.2.3** we try to give an all encompassing definition to the concepts concerning stronger forms of transitivity while studying correspondence between them. Using the sufficient condition **Theorem 2.3.1** we show that chaos implies that a set of points with a prime period of at least n is dense. In **Example 2.3.1** we give an counterexample of a system which is P_n dense but it is not Chaotic.In **Example 2.3.2** we consider a Chaotic Tent Map with P_3 dense .

Chapter 3

Sensitivity and its Stronger Forms

Chaos Theory is a mathematical theory used to study deterministic non-linear dynamical systems. It differentiates systems based on their sensitivity, which refers to their dependency on initial conditions. This chapter investigates the sensitivity of dynamical systems and their stronger forms. It covers various types of sensitivity, stronger forms of sensitivity, and the relationship between different types. It also discusses sensitivity for continuous maps on the famous [0,1] interval, subshifts. We also examine the relative strength of each sensitivity type.

3.1 Types of Sensitivities

In this section we talk about different types of sensitivities and then in the next section we will discuss examples of the types of sensitivities mentioned.

1. *Syndetic Sensitivity*: f is syndetically sensitive if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is syndetic.

2. *Cofinite Sensitivity*: f is cofinitely sensitive if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is cofinite.

3. *Thick Sensitivity*: f is thickly sensitive if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is Thick.

4. *Thick Syndetic Sensitivity*: f is syndetically sensitive if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is Syndetic and Thick.

5. *Thick Periodic Sensitivity*: f is Thickly Periodic sensitive if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is Thick and Periodic. [21]

6. *Strong Sensitivity*: f is Strongly Sensitive if there exists $\delta > 0$ such that for each x $\in$ X and each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$,there is a $y \in X$ with $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$.

7. *Asymptotic Sensitivity* : f is Asymptomatic Sensitive if there exists $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there exists $y \in X$ such that $d(x, y) < \epsilon$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta$ .In this case, the pair (x, y) is called an asymptotic sensitive pair.

8. *Li–Yorke Sensitivity* : f is Li Yorke Sensitive if there exists δ >0 such that for each $x \in X$ and ϵ >0 there exists $y \in X$ with $d(x, y)$< ϵ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y) > \delta$ [15]

Examples

Example of Syndetical Sensitivity

The below given example below illustrates Syndetical Sensitivity. The reader should note that the example also helps us disprove the claim that thick sensitivity implies syndetic sensitivity.

Let $L_0 = \mathbb{L}_0 = 0, L_1 = \mathbb{L}_1 = 2$ and $L_n = 2^{L_1+\cdots+L_{n=1}}$, $\mathbb{L}_n = L_1 + L_2 + \cdots + L_n$

for all $n \geqslant 2$ and

$X = [0,1] \cup (\bigcup_{n=1}^{\infty}[\mathbb{L}_{2n-1}, \mathbb{L}_{2n-1} + 2n - 1]) \cup (\bigcup_{n=1}^{\infty}[\mathbb{L}_{2n}, \mathbb{L}_{2n} + \frac{1}{2n}])$

$$f(x) = \begin{cases} x+2 & x \in [0,1] \\ \\ \frac{1}{2n(2n-1)}(x - \mathbb{L}_{2n-1} + \mathbb{L}_{2n}) & \text{if } x \in [\mathbb{L}_{2n-1}, \mathbb{L}_{2n-1} + 2n - 1] \\ & \text{for some } n \in \mathbb{N} \\ \\ (2n)(2n+1)(x - \mathbb{L}_{2n} + \mathbb{L}_{2n+1}) & \text{if } x \in [\mathbb{L}_{2n}, \mathbb{L}_{2n} + \frac{1}{2n}] \\ & \text{for some } n \in \mathbb{N} \end{cases}$$

For any nonempty open subset U of X, there exist a non-degenerate closed interval $I \subset [0,1]$ and $k \in \mathbb{N}_0$ such that $f^k(I) \subset U$. Thus, for any $n > k$,one has $\operatorname{diam}(f^{2n+1-k}(U)) \geqslant \operatorname{diam}(f^{2n+1}(I)) = (2n+1) \cdot |I| \to +\infty \ (n \to +\infty)$.
This implies that (X, f) is syndetically sensitive. [22]

Example of Cofinite Sensitivity

The example below illustrates Cofinite Sensitivity. The reader should note that the example also helps us disprove the claim that thick sensitivity implies syndetic sensitivity. $< X \times Y, f \times g >$ is multi-sensitive if and only if $< X, f >$ or $< Y, g >$ is multi- sensitive.

Let $L_0 = \mathbb{L}_0 = 0, L_1 = \mathbb{L}_1 = 2,$ and $L_n = 2^{L_1+\cdots+L_{n-1}} \cdot (2n)$,
$\mathrm{L}_n = L_1 + L_2 + \cdots + L_n$ for all $n \geqslant 2$, and set

$$X = [0,1] \cup \left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-2}}} [\mathbb{L}_{2n-1} + i, \mathbb{L}_{2n-1} + i + \frac{1}{2n}] \right) \bigcup$$
$$\left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-1}}} [\mathbb{L}_{2n} + (2n+1)i, \mathbb{L}_{2n} + (2n+1)i + 2n] \right)$$

$$Y = [0,1] \cup \left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-2}}+4n-4} [\mathbb{L}_{2n-1} + 2ni, \mathbb{L}_{2n-1} + 2ni + 2n - 1] \right) \bigcup$$
$$\left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-1}}-4n+2} [\mathbb{L}_{2n} + i, \mathbb{L}_{2n} + i + \frac{1}{2n}] \right)$$

For $n \in \mathbb{N}$ let

$\mathbb{A}_n = \left[\mathbb{L}_{2n-1} + 2^{\mathbb{L}_{2n-2}}, \mathbb{L}_{2n-1} + 2^{\mathbb{L}_{2n-2}} + \frac{1}{2n-1}\right]$,

$\mathbb{B}_n = \left[\mathbb{L}_{2n} + (2n+1) \cdot 2^{\mathbb{L}_{2n-1}}, \mathbb{L}_{2n} + (2n+1) \cdot 2^{\mathbb{L}_{2n-1}} + 2n\right]$,

$\mathbb{C}_n = \left[\mathbb{L}_{2n-1} + 2n \cdot (2^{\mathbb{L}_{2n2}} + 4n - 4), \mathbb{L}_{2n-1} + 2n \cdot (2^{\mathbb{L}_{2n2}} + 4n - 4)(2n-1)\right]$,

$\mathbb{D}_n = \left[\mathbb{L}_{2n} + 2^{\mathbb{L}_{2n-1}} - 4n + 2, \mathbb{L}_{2n} + 2^{\mathbb{L}_{2n-1}} - 4n + 2 + \frac{1}{2n}\right]$.

Define $f : X \to X$ and $g : Y \to Y$ respectively by

$$f(x) = \begin{cases} \frac{1}{2}x + 2 & \text{if } x \in [0,1] \\ x + 1 & \text{if } x \in [\mathbb{L}_{2n-1} + i, \\ & \mathbb{L}_{2n-1} + i + \frac{1}{2n}] \\ & \text{for some } 0 \leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-2}}\ , n \in \mathbb{N} \\ x + 2n + 1 & \text{if } x \in [\mathbb{L}_{2n} + (2n+1)i, \\ & \mathbb{L}_{2n} + (2n+1)i + 2n] \\ & \text{for some } 0 \leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-1}}\ , n \in \mathbb{N} \\ 2n(2n-1)(x - \mathbb{L}_{2n-1} - 2^{\mathbb{L}_{2n-2}}) + \mathbb{L}_{2n} & \text{if } x \in \mathbb{A}_n\ , n \in \mathbb{N} \\ \frac{1}{(2n)(2n+1)}(x - \mathbb{L}_{2n} - (2n+1) \cdot 2^{\mathbb{L}_{2n-1}}) + \mathbb{L}_{2n+1} & \text{if } x \in \mathbb{B}_n, n \in \mathbb{N} \end{cases}$$

and

$$g(x)=\begin{cases} x+2 & \text{if } x\in[0,1] \\ x+1 & \text{if } x\in[\mathbb{L}_{2n}+i, \\ & \mathbb{L}_{2n}+i+\frac{1}{2n}] \\ & \text{for some } 0\leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-1}}\ , n\in\mathbb{N} \\ x+2n & \text{if } x\in[\mathbb{L}_{2n}+2ni, \\ & \mathbb{L}_{2n}+2ni+(2n-1)] \\ & \text{for some } 0\leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-2}}\ , n\in\mathbb{N} \\ \frac{1}{2n(2n-1)}(x-\mathbb{L}_{2n-1}-2n\cdot(2^{\mathbb{L}_{2n-2}}+4n-4)+\mathbb{L}_{2n} & \text{if } x\in\mathbb{C}_n\ , n\in\mathbb{N} \\ \frac{1}{(2n)(2n+1)}(x-\mathbb{L}_{2n}-2^{\mathbb{L}_{2n-1}}+4n-2)+\mathbb{L}_{2n+1} & \text{if } \ x\in\mathbb{D}_n, n\in\mathbb{N} \end{cases}$$

$(X\times Y, f\times g)$ is cofinitely sensitive [22]

Example of Thick Sensitivity

The example below illustrates thick sensitivity.

cm Let $L_0=\mathbb{L}_0=0, L_1=\mathbb{L}_1=2,$ and $L_n=2^{L_1+\cdots+L_{n=1}}\cdot(2n),$

$\mathrm{L}_n=L_1+L_2+\cdots+L_n$ for all $n\geqslant 2$, and set

$$X = [0,1] \cup \left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-2}}} [\mathbb{L}_{2n-1} + i, \mathbb{L}_{2n-1} + i + \frac{1}{2n}] \right) \bigcup$$
$$\left(\bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{2^{\mathbb{L}_{2n-1}}} [\mathbb{L}_{2n} + (2n+1)i, \mathbb{L}_{2n} + (2n+1)i + 2n] \right)$$

For $n \in \mathbb{N}$ let

$\mathbb{A}_n = \left[\mathbb{L}_{2n-1} + 2^{\mathbb{L}_{2n2}}, \mathbb{L}_{2n-1} + 2^{\mathbb{L}_{2n-2}} + \frac{1}{2n-1}\right],$

$\mathbb{B}_n = \left[\mathbb{L}_{2n} + (2n+1) \cdot 2^{\mathbb{L}_{2n-1}}, \mathbb{L}_{2n} + (2n+1) \cdot 2^{\mathbb{L}_{2n-1}} + 2n\right],$

Define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{2}x + 2 & \text{if } x \in [0,1] \\ x + 1 & \text{if } x \in [\mathbb{L}_{2n-1} + i, \\ & \mathbb{L}_{2n-1} + i + \frac{1}{2n}] \\ & \text{for some } 0 \leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-2}}, n \in \mathbb{N} \\ x + 2n + 1 & \text{if } x \in [\mathbb{L}_{2n} + (2n+1)i, \\ & \mathbb{L}_{2n} + (2n+1)i + 2n] \\ & \text{for some } 0 \leqslant i \\ & \leqslant 2^{\mathbb{L}_{2n-1}}, n \in \mathbb{N} \\ 2n(2n-1)(x - \mathbb{L}_{2n-1} - 2^{\mathbb{L}_{2n-2}}) + \mathbb{L}_{2n} & \text{if } x \in \mathbb{A}_n, n \in \mathbb{N} \\ \frac{1}{(2n)(2n+1)}(x - \mathbb{L}_{2n} - (2n+1) \cdot 2^{\mathbb{L}_{2n-1}}) + \mathbb{L}_{2n+1} & \text{if } x \in \mathbb{B}_n, n \in \mathbb{N} \end{cases}$$

(X, f) is Thickly Sensitive [22]

Example of Strong Sensitivity

The example below illustrates Strong Sensitivity. It pertains to the Shift Map, which is widely used in Mathematics.

Let $\Sigma 2$ denote the space of all infinite sequences of 0's and 1's with the metric d(x,y)

$= \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$

Let X be the collection of all sequences which are eventually zero. Then, the shift map σ : X → X defined as $(\sigma(x))n = xn + 1$ is strongly sensitive.

However, as orbits of any two points eventually coincide, the map fails to be asymptotically sensitive.

Example of Asymptomatic Sensitivity

The below-given example illustrates Asymptomatic Sensitivity .

Let $X = [1, \infty)$. Define a map $f : X \to X$ as $f(x) = x^2$.

Then the map defined is asymptotic sensitive but fails to be Li – Yorke sensitive. [24]
Example of Li-Yorke Sensitivity

The below given example illustrates Li Yorke sensitivity.The reader should note that it is not transitive.

Let us consider a dynamical system which will be a two-sided subshift $X \subset \Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ with a special inner structure. For words u, v we will write $[u.v]$ to denote cylinder consisting of $x \in \Sigma_2$ such that $x_{[-s,0)} = u$ and $x_{[0,k)} = v$ for some $s, k > 0$.

Fix an increasing sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_0 = 1$ and $a_{n+1} > 2^{n+1} \sum_{i=1}^{n} a_n$.
Let $l : \mathbb{L}(\sum_2^+) \to \mathbb{N}$ be a function defined on all finite words in such a way that if

$w = w_0 \cdots w_{n-1} \in [L]_n(\Sigma_2)$ then $l(w) = \sum_{i=0}^{n-1} 2^i w_i$.

For example $l(01) = 2, l(011) = 6$, etc.

Define a map

$\pi : \sum_2^+ \to \sum_2$ by putting for each $x \in \Sigma_2^+ \to \Sigma_2$ and each $i \in \mathbb{Z}$:

$$\pi(x)_i = \begin{cases} 1, & l(x_{[0,n)}) = j \text{ and } i = a_j \text{ for some } n > 0 \\ \\ 0, & \text{otherwise} \end{cases}$$

It is not hard to verify that the map π is continuous. Note that each point $\pi(x)$ can have symbols 1 only at positions in the sequence $\{a_n\}_{n=1}^{\infty}$. Furthermore, if $x \neq y, x, y \in \Sigma_2^+$, then for some n and hence for $i \geqslant a_n$ we never have $\pi(x)_i = \pi(y)_i = 1$ because $l(x_{[0,i)}) \neq l(y_{[0,j)})$for every $i, j \geqslant n$

Put $X = \cup_{i \in \mathbb{Z}} \sigma^i(\pi(\Sigma_2^+))$ and observe that X is a two-sided subshift since it is not only invariant but also closed because ${}^{\infty}0^{\infty}, {}^{\infty}010^{\infty} \in X$.For any $y \in X$ there is a p $\in \mathbb{Z}$ such that $y \in \sigma^p(\pi(\sum_2^+))$If there are integers $i < j$ such that $y_i = y_j = 1$ then we claim that there exists exactly one n such that $a_n > j - i$ and $a_n - a_{n-1} \leqslant j - i$.By the construction, there are uniquely determined indexes $0 \leqslant m < n$ such that $a_m = i + p < j + p$. Observe that n satisfies $a_n = j + p > j + p - (i + p) = j - i$ since $i + p = a_m > 0$ and $a_n - a_{n-1} \leqslant a_n - a_m = j - i$.On the other hand if $t > n$ satisfies the conditions , then $a_t - a_{t-1} > (2^t - 1)a_{t-1} > (2^n - 1)a_n > j - i$, a contradiction.If $t < n$ satisfies the conditions, then $j - i = a_n - a_{n-1} \geqslant a_n - a_{n-1} \geqslant (2^n - 1)a_{n-1} \geqslant a_{n-1t}$ again a contradiction. Indeed, the claim holds. But then $\sigma^{j-a_n}(y) \in \pi(\Sigma_2^+$ and then there is exactly one $x \in \Sigma_2^+$ such that $\pi(x) = \sigma^{j-a_n}(y)$. Furthermore, for every $r \neq j - a_n$, we have $\sigma^r(y) \neq \pi(\Sigma_2^+)$

Hence(X, σ) is Li–Yorke sensitive [23]

3.2 Stronger Forms of Sensitivity

Lemma 9. *If (*A*) is an* M-*system, with* S *as a minimal subset of* A *and* V *as a nonempty open subset of* A*, then* N*(*V*,* B*(*S*,* δ*)) is thickly syndetic for any* $\delta > 0$.

Proof. For any $k \in Z^+$, since g is uniformly continuous, so $\exists\ \delta^{'} > 0$, such that for any $d(a_1, b_2) < \delta^{'}, d(g^i(a_1), g^i(a_2)) < \delta(i = 1, 2, ..., k)$. For any transitive point $a \in V$ and any minimal set $S, m \in Z^+$, such that $d(g^m(a), S)) < \delta'/2$,so there is a minimal point $\acute{a} \in V$ with $d(g^m(a), S)) < \delta^{'}$. Since $a^{'}$is almost periodic, there exists a syndetic set$n_i$with $d(g^{n_i}(a), S) < \delta^{'}$, so $d(g^{n_i+j}(\acute{a}), S) < \delta(j = 1, 2, ..., k, i = 1, 2, 3, ...)$. So we have $N(V, B(S, \delta))$ that is thickly syndetic.

Minimal System

A dynamical system $(A, g)|$ is called minimal if $A|$ does not contain any non-empty, proper, closed f-invariant subset. In such a case we also say that the map g itself is minimal. Thus, one cannot simplify the study of the dynamics of a minimal system by finding its nontrivial closed subsystems and studying first the dynamics restricted to them.

Given a point x in a system $(A, g), |Orb_g(a) = a, f(a), g^2(a), ...|$ denotes its orbit (by an orbit we mean a forward orbit even if $f|$ is a homeomorphism) and $\omega_g(a)|$ denotes its $\omega|$-limit set, i.e. the set of limit points of the sequence $a, g(a), g^2(a), ...|$. The following conditions are equivalent:

1. (A, g) is minimal, Every orbit is dense in A , $\omega_g(a) = A$ for every $a \in A$.

2. A minimal map g is necessarily surjective if A is assumed to be Hausdorff and compact.

In this section, we define a set $N(Y, \epsilon) = n \in N : diam((g)^n(Y)) > \epsilon$ we can tell for sensitivity for given dynamical system $(X, (g))$. That is $(A, (g))$ is sensitive if and only if $N(Y, \epsilon) \neq \phi$ for some $\epsilon > 0$ and $Y \neq \phi, Y \subset A$.

In the following theorems, we will see how fewer conditions can imply different types of sensitivity for the given topological dynamical system :

Theorem 3.2.1. If (A, G) is minimal and sensitive, then we can say (A, g) is syndetically sensitive.

Proof. As given g is sensitive hence, by definition of sensitivity for sensitivity constant ϵ $\exists$ a, b $\in Y \subset A$, and$Y \neq \phi$ and $n \in Z^+$ such that $d(g^n(a), g^n(b)) > \epsilon$. We can say $\exists$ an open set $Y_1 \subset Y$ such that $d(g^n(a), g^n(Y_1)) > \epsilon$. It is also mentioned that (A, g) is minimal, $\exists$ $k \in Z^+$ such that $g^k(a) \in Y_1 \subset Y$ so,
$d(g^n(a), g^{n+k}(a)) \geqslant d(g^n(a), g^n(Y_1)) > \epsilon$.

Because f is uniformly continuous we can write: $\exists\delta \in (0, \epsilon/4)$ such that for any $d(a_1, a_2) < \delta$, $d(g^i(a_1), g^i(a_2)) < \epsilon/4, (i = 1, 2, ..., k)$. Also, $g^n(a)$ is a minimal point, so $N((g^n(a), B(g^n(a), \delta))$ is syndetic. For any $m \in N(g^n(a), B(g^n(a), \delta))$, we have $d(g^n(a), g^{m+n}(a)) < \delta$, so $d(g^{k+n}(a), g^{k+n+m}(a)) < \epsilon/4$; that is,
$d(g^{m+n}(a), g^{m+n+k}(a)) \geqslant d(g^n(a), g^{k+n}(a))$,
d (g$^n(a), g^{m+n}(a)) - d(g^{k+n}(a), g^{k+m+n}(a)) > \epsilon/2$. [28]

This means $N(Y, \epsilon/2)$ is syndetic. □

Theorem 3.2.2. Weak mixing implies thick sensitivity.

Proof. Let $\epsilon \in 0, diam(A)$ and let V be a nonempty open set of A. [29] $\overline{g}$ is transitive

. Y is a fixed point of $\overline{g}(\overline{g}(A) = A)$. By density of transitive points of $\overline{g}$ in $K(Y)$, we have $K \subset V$, a transitive point of $\overline{g}$. [30] $N(K, B_H d(A, \epsilon))$ is thick; that is, $N(V, \epsilon)$ is thick, because $N(K,\ Y(A, \epsilon)) \subset N(V, \epsilon)$.

□

Theorem 3.2.3. If (A, g) is a minimal weakly mixing TDS, then it is thickly syndetically sensitive.

Proof. Since A is a fixed point of $(K(A), \overline{g})$, by the proof of [29]. If (A, g) is a transitive non-minimal TDS with a as a transitive point and S as a minimal set of A , then $N(a,\ Y(S, \delta))$ is thick for every $\delta > 0$, it is sufficient to prove that $n : H_d(g^n(V), A) < \epsilon$ is syndetic for any open set V of A and any $\epsilon > 0$. Given V and ϵ, let $K \subset V$ be a transitive point for $\overline{g}$. So there exist $n \in Z^+$ such that $H_d(g^n(K), A) < \frac{\epsilon}{6}$. Choose $\gamma > 0$ to be an $\frac{\epsilon}{6}$ modulus of uniform continuity for g^n so that $B(K, \gamma) \subset V$. Then $d(b_i, c_i) < \gamma$ for $i = 1, ..., k$ implies $b_i, ..., b_k \subset V$ and $H_d(g^n(a, b_1, ..., b_k), g^n(a, c_1, ..., c_k)) < \frac{\epsilon}{6}$.

Since the orbit of a is dense in A, there are strictly increasing integers $n_i : i = 1, 2, ..., k$ such that $d(g^n{}_i(a), c_i) < \gamma$ for $i = 1, ..., k$. Hence, $H_d(g^n(a, g_1^n(a), ..., g_k^n(a), A) < \frac{\epsilon}{2}$. Because g is uniformly continuous, so $\exists\ \delta \in (0, \frac{\epsilon}{4})$ such that for any $d(a_1, a_2) < \delta,\quad d(g^i(a_1), g^i(a_2)) < \frac{\epsilon}{2}$, $(i = 1, 2, ..., n_k)$.Since $g^n(a)$ is a minimal point,$N(g^n(a), B(g^n(a), \delta))$ is syndetic.

For any $m \in N(g^n(a), B(g^n(a), \delta))$ i.e. $d(g^{m+n}(a), g^n(a)) < \delta)$, we have $d(g^m(g^{n+n_i}(a)), g^{n+n_i}(a)) = (g^{n_i}(g^{m+n}(a)), g^{n_i}(g^n(a))) < \epsilon/2$, that is, $H_d(g^{m+n}(A_k), g^n(A_k)) < \frac{\epsilon}{2}$. Then, $H_d(g^{m+n}(V), A) \leqslant H_d(g^{m+n}(A_k), A) \leqslant H_d(g^{m+n}(A_k), g^n(A_k)) + H_d(g^n(A_k), A) < \epsilon$.

□

Theorem 3.2.4. If (A, g) is a non-minimal M -system, then it is thickly syndetically sensitive.

Proof. Let A', B' be minimal sets of g with $d(A', B') = a'$ and let V be a nonempty open set of A. For any $k \in Z^+$, since g is uniformly continuous, so $\exists\ \delta > 0$, such that for any $d(a_1, b_2) < \delta, d(g^i(a_1), g^i(a_2)) < \frac{a'}{4}$, $(i = 1, 2, ..., k)$.

By Lemma 1, we have $N(V, B(A', \delta)), N(V, B(B', \delta))$, thickly syndetic, so we have $N(V, B(A', \delta)), N(V, B(B', \delta))$, syndetic. Thus, for every $m \in N(V, B(A', \delta))$, N(V, B(B', δ)) , $\{m, m + 1, ..., m + k\} \subset N(V, B(A', \frac{a'}{4})) \cap N(V, B(B', \frac{a'}{4}))$. By the arbitrary nature of k ,$N(V, B(A', \frac{a'}{4})) \cap N(V, B(B', \frac{a'}{4}))$ is thickly syndetic; that is, $N(V, \frac{a'}{2})$ is thickly syndetic.

Corollary 0.4. *Devaney chaos (or P-system without isolated points) implies thickly periodic sensitivity.*

Proof. The proof is similar to the above theorem. □

Deriving sensitivity from transitivity

Stronger forms of transitivity along with few special conditions imply stronger forms of sensitivity. We will now focus on results that could be deduced in the context of stronger forms of transitivity.

Proposition 4. *Let (A, g) be a mixing dynamical system. Then, for any positive $\delta < diam(A), f$ is cofinitely sensitive with δ as a constant of sensitivity.*

Devaney defined a dynamical system (A, g) to be chaotic if g is transitive, and sensitive and if $P(g)$ is dense in A. [14] It was soon observed that if A is not finite, g is transitive and if $P(g)$ is dense in A, then g is sensitive. [25] This was improved by showing that if $g$is a transitive, non-minimal map with a dense set of minimal points, then f is sensitive. [32]

Theorem 3.2.5. Let (A, g) be a dynamical system. If g is syndetically transitive but not minimal, then g is syndetically sensitive.

Proof. Let $a' \in A$ be such that $O_g(a')$ is not dense in A. Let $b' \in A\ \overline{O_g(a')}$, and put $\delta = \frac{1}{4}\ , d(b', \overline{O_g(a')}) > 0$. Write $V = B(b', \delta)$. If $Y \subset A$ is any nonempty open set, then $N_g(Y, V)$ is syndetic, with say M_1 as a bound for the gaps. Choose an open set W around a such that $a \in W \Rightarrow d(g^i(a'), g^i(a)) < \delta$ for $i = 0, 1, ..., M_1$. Note that then $d(g^i(W), V) \geqslant 2\delta$ for $i = 0, 1, ..., M_1$, by the choice of δ. Now, $N_g(Y, W)$ is also syndetic.

Let M_2 be a bound for the gaps in $N_g(Y, W)$. We show that $N_g(Y, \delta)$ is syndetic with $M_1 + M_2$ as a bound for the gaps. Let $n \in N$. Choose $j \in 1, ..., M_2$ and $u \in Y$ such that $g^{n+j}(u) \in W$. Then, by the choice of W, one has that for every $i = 1, ..., M_1, d(g^{n+j+i}(u), V) \geqslant 2\delta$. Choose $i = 1, ..., M_1$ and $u' \in Y$ such that $g^{n+j+i}(u) \in V$. Then, for this particular i, we have $d(g^{n+j+i}(u), g^{n+j+i}(u)) \geqslant 2\delta > \delta$. Since $n \in N$ is arbitrary and since $j + i \leqslant M_1 + M_2$, the argument is complete. □

Corollary 0.5. *For a syndetically transitive dynamical system, sensitivity implies syndetical sensitivity.*

Proof. If the system doesn't have a dense set of minimal points, we apply the above theorem. □

Example 3.2.1. The example illustrates Syndetical Transitivity alone doesn't imply sensitivity.

If α is irrational, then the isometry $a \to e^{2\pi i\alpha}a$ on the unit circle is known as an irrational rotation. It is well known that any irrational rotation is minimal, and hence syndetically transitive. This example shows that syndetical transitivity alone cannot imply sensitivity. However, we have the following sufficient condition as our next proposition:

Proposition 5. *Let*(Y, g) *be a dynamical system. If* g *is syndetically transitive and if* $\inf_{n \in A'} \sup_{a \in A} d(a, g^n(a)) > \delta > 0$ *for some thick set* $A' \subset N$*, then* g *is syndetically sensitive with* δ *as a constant of sensitivity.*

Proof. Let $Y \subset A$ be nonempty open. Since $N_g(Y, Y)$ is syndetic and A' is thick, there exists $n \in N_g(Y, Y) \cap A'$. Put $W = Y \cap g^{-n}(Y)$. Then, W is nonempty and open. Since $n \in A'$, by hypothesis, we can find $a \in A$ such that $d(a, g^n(a)) > \delta$. Let V be an open set containing a such that $b \in V$ implies $d(b, g^n(b)) > \delta$. Now, consider the syndetic set $N_g(W, V)$. We claim that it is contained in $N_g(Y, \delta)$. Let $k \in N_g(W, V)$ and let $a' \in W \subset Y$ be such that $g^k(a') \in V$. Then, $d(g^k(a'), g^{k+n(a')}) > \delta$ by the choice of V . So if we put $b' = g^n(a')$, then $d(g^k(a'), g^k(b')) > \delta$. Also, $b' \in g^n(W) \subset Y$. Therefore $k \in N_g(Y, \delta)$, and this establishes the claim. □

Corollary 0.6. *Let* (A, g) *be a syndetically transitive system. Suppose that there exist two distinct points* $a, b \in A$ *and a thick set* $n_k : k \in N$ *with* $\lim_{k\to\infty} d(g_k^n(a), g_k^n(b)) = 0$*. Then* g *is syndetically sensitive.*

Proof. Choose a positive $\delta < 1/3d(a, b)$. Let $k_0 \in N$ be such that $d(g_k^n(a), g_k^n(b)) < \delta$ for every $k \geqslant k_0$. Then, for each $k \geqslant k_0$, by triangle inequality we have that $d(a, g_k^n(a)) > \delta$ or $d(b, g_k^n(b)) > \delta$. So the above proposition applies with $A' = n_k \in N : k \geqslant k_0$. □

Sensitivity of continuous maps on [0,1]

We'll learn about how every sensitive map $g : [0, 1] \to [0, 1]$ is cofinitely sensitive. It is partially because of the ample presence of the periodic points in the given set .

In the following theorems and prepositions, we will repeatedly use the following concept: if $g : [0, 1] \to [0, 1]$ is sensitive, then for every interval $J \subset [0, 1]$ and every $n \in N$, $g^n(J)$ is also an interval.

Lemma 10. *Let $L \subset \mathbb{R}$ be a compact interval and $g : L \to L$ be sensitive. Then, the closure of the set of periodic points of g contains an interval.*

Proof. From proposition 2.2.5 (Blokh), it can be deduced that there exists a closed interval $J \subset L$ and an $n \in N$ such that J is $g - n$-invariant and $g^n|J : J \to J$ is transitive. of [31] But a transitive map of a closed interval has a dense set of periodic points [45]. Thus $J \subset \overline{P(g^n)} = \overline{P(g)}$. □

Theorem 3.2.6. Let $g : [0, 1] \to [0, 1]$ be sensitive. Then g is cofinitely sensitive.

Proof. Let $\delta > 0$ be a constant of sensitivity for f. Choose finitely many periodic points $a_1, ..., a_r \in [0, 1]$ such that for any interval $J \subset P(g)$ with $diam[J] > \delta$, we have that $|J \cap a_1, ..., a_r| \geqslant 2$. Let $\alpha = min|a_i a_j| : 1 \leqslant i < j \leqslant r > 0$ and let $k \in N$ be such that $g^k(x_i) = a_i$ for $1 \leqslant i \leqslant r$. Let $\beta > 0$ be such that for every $a, b \in [0, 1], |ab| \leqslant \beta$ implies $|g^i(a)g^i(b)| < \alpha$ for $0 \leqslant i \leqslant k$. We claim that f is cofinitely sensitive with β as a constant of sensitivity. □

Let $J \subset [0, 1]$ be any interval. Since $diam[g^n(J)] > \delta$ for infinitely many n, it is easy to see that $\cup g^n(J)$ has only finitely many connected components. Hence the same is true for $\bigcup_{n=0}^{\infty} g^n(J)$. Therefore, one can find a connected component L, which must be a closed interval, of $\overset{\infty}{\underset{n=0}{\cup}} g^n(J)$ and an $n \in N$ such that $g^n(L) \subset L$. Then, $g^n|L : L \to L$ is sensitive (with some constant of sensitivity). By the above lemma, $P(g^n|L)$ contains an interval. This implies that, for some $s \in N, g^s(J) \cap P(f)$ contains an interval, say K. Now, for some $t \in N$, $diam[g^t(K)] > \delta$. But the interval $g^t(K)$ is contained in $P(g)$ as $P(g)$ is g -invariant. Therefore, $|g'(K) \cap a_1, ..., a_r| \geqslant 2$.

As a consequence, $diam[g^{t+j}(K)] > \beta$ for every $j \in N$, by the choice of β. Hence, $diam[g^{s+t+j}(J)] > \beta$ for every $j \in N$. That is, $s + t + N \subset N_g(J, \beta)$. □

Thus all sensitive maps of $[0, 1]$ exhibit a very strong form of sensitivity. We know that transitivity implies sensitivity on [0, 1]. [45] Therefore, by the above theorem, all transitive maps on $[0, 1]$ are cofinitely sensitive. In the rest of the section, we distinguish sensitivity, syndetical sensitivity, and cofinite sensitivity using subclasses of dynamical systems known as *subshifts.*

3.3 Relationship Between Different Types of Sensitivity

In plain words, sensitivity simply means that given any point, there exists another point arbitrarily close such that the orbits of these two points move apart by a fixed distance after some finite instants. The system is strongly sensitive if after a particular instant, for each successive instant, there are points arbitrarily close to any point, such that their orbits move apart by a fixed distance from the orbit of this particular point. If for any point there is a point arbitrarily close by, such that the orbits of these two points move apart infinitely often, then the system is asymptotically sensitive. If in addition to moving apart infinitely often, these orbits also come arbitrarily closer infinitely often, then the system is Li–Yorke sensitive. In general, these properties though distinct, satisfy the relation:

sensitive $\Leftarrow$ strongly sensitive

sensitive $\Leftarrow$ asymptotic sensitive $\Leftarrow$ Li Yorke sensitive. [24]

Every strongly mixing semiflow is weakly mixing. Every strongly multisensitive semiflow is multi-sensitive. Every thickly syndetically sensitive semiflow is thickly sensitive. Every thickly syndetically sensitive semiflow is syndetically sensitive. Every thickly periodically sensitive semiflow is periodically sensitive. Every thickly periodically sensitive semiflow is thickly syndetically sensitive. Every periodically sensitive semiflow is syndetically sensitive. Every multi sensitive (resp. thickly sensitive; syndetically sensitive) semiflow is sensitive. [33]

Moreover, we can infer from previous definitions that mixing $\Rightarrow$ thickly periodically sensitive $\Rightarrow$ thickly syndetical sensitivity $\Rightarrow$ thick sensitivity and syndetical sensitivity:

Conclusion

In this chapter we studied various forms of sensitivity. We analysed interesting examples arising from concrete spaces which displayed one or more type of sensitivity. Then we categorized these sensitivities based on their relative strength. **Theorem 3.2.2** and **Theorem 3.2.3** helped us establish a relationship between Weak mixing and Thick sensitivity.We also proved **Theorem 3.2.5**, which helps us deduce nature of sensitivity in the context of stronger forms of transitivity. We also proved **Theorem 3.2.6** which helps establish the fact that every continuous map on [0,1] is cofinitely sensitive. Using all the necessary and sufficient conditions that we proved in this chapter we established correspondence between various forms of sensitivity .

Chapter 4

Chaos Theory

Chaos, by definition, refers to the unpredictability of dynamical systems over time. It is one of the most important characteristics defining dynamical systems. in this section, we study how chaos is defined, what it implies and what are various examples of it. We also provide an overview of relations between different types of chaos. Finally, We study some interesting examples of chaos like the 'tent map', the 'logistic map', and the 'shift map'. We also observe in these, the 'near chaos' cases, i.e., where these are not chaotic due to one property not being satisfied.

4.1 Different Types of Chaos

Here, we provide an overview of different Definitions of chaos given by different mathematicians:

Definition 4.1.1 (Devaney's Chaos [40])**.** A cascade (Y, f) is called *chaotic* if it satisfies:

1. (Y, f) is topologically transitive

2. (Y, f) is sensitive

3. $Per(Y)$ is dense in Y or $\overline{Per(Y)} = Y$

Definition 4.1.2 (Wiggins Chaos [41]). Let $f : Y \to Y$ be a continuous map where Y is a metric space. Then f is said to be *Wiggins Chaotic* if:

1. f is topologically transitive.

2. f depicts sensitive dependence on initial conditions.

Definition 4.1.3 (Li-Yorke Chaos [42]). Under this condition, let $x, y \in Y$. The pair $(x, y) \in (Y, Y)$ is a Li-Yorke scrambled pair if:

1.
$$\lim_{n\to\infty} sup(d(f^n(x).f^n(y))) > 0$$

2.
$$\lim_{n\to\infty} inf(d(f^n(x).f^n(y))) = 0$$

The map is *Li-Yorke chaotic* if it has an uncountable scrambled set in Y.

Definition 4.1.4 (Lyapunov Chaos [43]). Consider a continuous differentiable map $f : \mathbb{R} \to \mathbb{R}$. Then f is said to be *Lyapunov Chaotic* if:

1. f is topologically transitive

2. f has a positive Lyapunov constant

Definition 4.1.5 (Knudsen Chaos [44]). let $f : Y \to Y$ be a continuous map on a metric space (y, d), then the dynamical system (Y, f) is *Knudsen Chaotic* if:

1. f has dense orbits.

2. f is sensitive to initial conditions.

Definition 4.1.6 (Martelli Chaos). Let f be a continuous map from a compact metric space X with standard metric d into itself. Then f is said to be *Martelli Chaotic* if there exists $x_0 \in X$ such that x_0 has a dense orbit that is unstable.

Definition 4.1.7 (Auslander-Yorke Chaos [52])**.** Let X be a space with usual metric d, then a map f is *Auslander-Yorke chaotic* if it is sensitive and topologically transitive. A semiflow (T, X) is said to be strongly Auslander-Yorke chaotic if it is topologically transitive, sensitive, and has a dense set of Poisson stable points.

Definition 4.1.8 (Ruelle Takens Chaos [52])**.** Let X be a space with usual metric d,then a map f is *Ruelle-Takens chaotic* if it is point-transitive and sensitive. A semiflow (T, X) is said to be *strongly Ruelle-Takens chaotic* if it is point-transitive, sensitive, and has a dense set of Poisson stable points.

Definition 4.1.9 (Poincaré Chaos [53])**.** A semiflow (T, X) is said to be *Poincaré Chaotic* if it has an unpredictable transitive point.

Theorem 4.1.1 ([49])**.** Assume that for a metric space X, a map $f : X \to X$ is Devaney chaotic. Then it has an uncountable scrambled set for f and hence it has Li-Yorke chaos. Moreover, if f is totally transitive, then f is densely Li-Yorke chaotic. Particularly, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.

Proof. Assume that f has a fixed point p. Thus, since f is transitive, for every $x \in X$, $PR(x)$ is a G_δ. Denote by $Tran_f$, the set of transitive points of f. If $x \in Tran_f$, there exists an n_i with $f^{n_i}(x) \to p$. This implies that $f^i(x)$ is proximal to x for each $i \geqslant 1$. Hence, $PR(x)$ is a dense G_δ set for each $x \in Tran_f$.
For further proof, we require a lemma:

Lemma 11. *Assume that X is a complete separable metric space without isolated points. If R is a symmetric relation with the property that there is a dense G_δ subset A of X such that for each $x \in A$, $R(x)$ contains a dense G_δ subset, then there is a dense, subset B of X with uncountably many points such that $B \times B \backslash \Delta \subset R$*

Now, let $R = LYR(X, f)$ and $A = Tran_f$. For each $x \in A$, it is clear that $R(x) = PR(x) \backslash AR(x)$ contains a dense G_δ subset. By the above lemma, there is a subset B of X such that $B \times B \backslash \Delta \subset R$ and B is uncountable. Clearly, B is a scrambled set of f.

Moreover, assume that f has a periodic point of period $n > 1$. Let $x \in Tran_f$. Then $\omega(x, f) = X$. Set $D_i = \omega(f^i x), f^n)$ for each $0 \leqslant i \leqslant n-1$. As $f(D_i) = D_{i+1(\mod n)}$, we know that each D_i is uncountable and contains a periodic point of f with period n. As $f^n|_{D_0}$ is transitive, and has a fixed point of f^n, we can use the result just proved. So, there is an uncountable scrambled set B for f^n. Clearly, B is also a scrambled set for f. Hence, f is Li-Yorke Chaotic. Following this, if ff is totally transitive, a similar argument shows that f is densely Li-Yorke chaotic. Hence proved. □

Theorem 4.1.2. Devaney Chaos implies Wiggins Chaos and Martelli Chaos directly.

Proof. Now, since Wiggins chaos demands only 2 or 3 criteria required for Devaney chaos, hence, Wiggins chaos is directly implied by Devaney chaos.

As for Martelli chaos, for any point $x_0 \in X$, observe that it is dense with r equal to the sensitivity level as f is already Devaney chaotic. Thus, Devaney chaos implies Martelli's chaos as well. Hence proved. □

4.2 Theorems on Chaos

Now, we will see some theorems on Chaos.

Theorem 4.2.1. [44] If X is a metric space and $X \to X$ is transitive and has dense periodic points, then it is sensitive as well. Hence, f is chaotic.

Proof. Since the periodic points are dense, observe that we can find a $\delta > 0$ such that for any $x \in X$, there is a periodic point q such that $d(x, O(q)) > \delta/2$ where $O(q)$ denotes the orbit of q and d is the distance function on X. Similarly, we can find 2 periodic points q_1 and q_2 such that $d(O(q_1), O(q_2)) = \delta$. Hence, for any $x \in X$ let

$d_1 = d(x, O(q_i)); i = 1$ *or* 2, be distance from the farther orbit, and by the triangle inequality

$$d(x, O(q_j)) \geqslant d(x, O(q_i)) + d(O(q_i), O(q_j)) \geqslant \delta/2; j \neq i$$

so, any x is at least a distance of at least $\delta/2$ from any of the 2 orbits. Now, we will prove sensitivity.

Let $x \in X$ be arbitrary and let $\mathcal{N}(x)$ be a neighborhood of x. Let $B_\delta(x)$ be a ball with radius δ centered at x. As periodic points are dense, we can find a $p \in N \cap B_\delta(x)$ with a period, say n. From the above, we can find a periodic point $q \in X$ with orbit $O(q)$ at a distance of at least 4δ from x. Also, let

$$M = \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q)))$$

Since M is the inverse of an open ball, it is open. Also, it is non-empty since $q \in M$. Also, by transitivity, there is a $y \in U$ and a natural number l such that $f^l(y) \in M$. Now, let j be the integral part of $l/n + 1$. So, $1 \leqslant nj - l \leqslant n$. By construction,

$$f^{nj}(y) = f^{nj-l}(f^l(y)) \in f^{nj-l}(M) \subseteq B_\delta(f^{nj-l}(q))$$

and since $f^{nj}(p) = p$.Using triangle inequality:

$$d(f^{nj}(p), f^{nj}(y)) = d(p, f^{nj}(y)) \geqslant d(x, f^{nj}(y)) - d(p, x) \geqslant$$

$$d(x, f^{nj-l}(q)) - d(f^{nj-l}(q), f^{nj}(y)) - d(p, x)$$

Finally, since $p \in B_\delta(x)$ and $f^{nj}(y) \in B_\delta(f^{nj-l}(q))$, we have:

$$d(f^{nj}(p), f^{nj}(y)) > 4\delta - \delta - \delta = 2\delta$$

Using triangle inequality again,

$$d(f^{nj}(p), f^{nj}(y)) \geqslant d(f^{nj}(x), f^{nj}(y)) - d(f^{nj}(p), f^{nj}(x)) \geqslant 2\delta - \delta = \delta$$

A similar argument can be made for $d(f^{nj}(p), f^{nj}(x))$. Either way, we have found a point in N such that its nj^{th} iterate makes the distance of a point x from p or y more than a number δ. $\square$

Theorem 4.2.2 ([25, 45]). Let I be a, not necessarily finite, interval and $f : I \to I$ be a continuous and topologically transitive map. Then the map is chaotic. That is, periodic points of f are dense in I and f has sensitive dependence on initial conditions.

Before proving this theorem, we need to prove a lemma:

Lemma 12. *Suppose that I is a, not necessarily finite, interval and $f : I \to I$ is a continuous map. If $J \subset I$ is an interval which contains no periodic points of f and let $x, f^m(x), f^n(x) \in J$ for $0 < m < n$. Then either $x < f^m(x) < f^n(x)$ or $x > f^m(x) > f^n(x)$.*

Proof. We will prove this by contradiction. Suppose we can find a $x \in J$ such that $x < f^m(x)$ and $f^m(x) > f^n(x)$.

Let

$$h(x) = f^m(x)$$

. Then, from our assumption, $x < h(x)$. Now, for any $k \in \mathbb{N}$ such that $k \geqslant 1$, let $h^{k+1}(x) < h(x)$. Then function $h^k(x) - x$ is positive at x and negative at $g(x)$. By Intermediate Value Theorem, there exists a point $c \in (x, g(x)) \subset J$ such that $g^k(c) - c = 0$, giving a km-periodic point of f in J. Hence, $x < g^k(x)$ for all $K \geqslant 1$. In particular, for $k = n - m > 0$, we have $x < g^{m-m}(x) = f^{(n-m)m}(x)$. Since we assumed that $f^{(n-m)}(f^m(x)) < f^m(x)$, taking $g = f^{n-m}$, we get $f^{(n-m)m}(f^m(x)) < f^m(x)$.

But then again, using the above argument, we get an $(n-m)m$-periodic point in J and thus we have a contradiction. So, our lemma is proved. A similar argument can be made for the other case $(x > f^m(x) > f^n(x))$ $\square$

Now, we prove the theorem:

Proof of Theorem 4.2. Suppose the f is continuous and topologically transitive. Due to the above result, we only need to show that periodic points of I are dense in I. This is because, by *Theorem 4.1*, transitivity and denseness of periodic points imply chaos.

We will prove this using contradiction. Let periodic points be not dense in I. Then, there exists an interval $J \subset I$ with no periodic points. Let $x \in J$ be such that it is not an endpoint of J. Moreover, we take an open neighborhood of x, $\mathcal{N} \subsetneq I$, and an open interval $E \subset J \backslash N$. Since f is topologically transitive on I, there exists a natural number $m > 0$ with $f^m(\mathcal{N}) \cap E \neq \emptyset$ and thus a $y \in J$ with $f^m(y) \in E \subset J$. Now, since J contains no periodic points, we know that $y \neq f^m(y)$ and since f is continuous, this implies that we can find a neighborhood U of y with $f^m(U) \cap U = \emptyset$. Now, since U is an open set, topological transitivity states that we can find an $n > m$ and $z \in U$ with $f^n(z) \in U$. But then we have $0 < m < n$ and $z, f^n(z) \in U$ which violates the lemma. Thus we have a contradiction and our theorem is proved. $\square$

4.3 Illustrations of Chaos on Different Dynamical Systems

In this section, we provide various examples of the results mentioned above. These examples further strengthen our theorems and proofs mentioned above.

Example 4.3.1. Define a *tent map* f on metric $X = [0, 1]$ with standard topology as :

$$f(x) = \min\{cx, c(1-x)\}$$

for $c \in \mathbb{N}$ We will show that for $c > 1$, this is chaotic, not only in the sense of Devaney but in the sense of Lyapunov as well.

Proof. Now, for $c < 1$, observe that for any $x \in X$, $f^n(x) < 1$. This is because if for some k, without loss of generality, let $f^k(x) = cf^{k-1}(x) > 1$, then $f^{k-1}(x) > \frac{1}{c} > 1$ and so, we trace back to x where $x > \frac{1}{c \times (1-c) \times \cdots} > 1$ which is a contradiction. Thus, for any sensitivity level $\beta > 1$, we cannot find any $y \in X$ such that $d(x, y) > 1$ since both x and y are less than 1, hence the tent map is not sensitive for $c < 1$ and hence not Devaney chaotic.

Using [38], for $c > 1$, let A and B be 2 non-empty open subsets of X, we will prove that there exists an $n \in \mathbb{N}$ such that $nA = X$ and hence $f^n(A) \cap B \neq \varnothing$.

Now, observe that the map f^n maps any subinterval of form, without loss of generality, $J_{k,n} = [\frac{k}{c^n}, \frac{k+1}{c^n}]$; for $k = 0, 1, 2, \ldots, c^n - 1$. And for this particular n, successive iterations make these subintervals coincide such that $f^n(j_{k,n}) = [0, 1]$.So, for any $A \subset X$, we can find an n sufficiently large such that $J_{k,n} \subseteq A$ and

$$[0, 1] = f^n(J_{k,n}) \subseteq f^n(A)$$

hence

$$X = f^n(A)$$

hence X is the orbit of A, thus, there is an $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \varnothing$. So, f is topologically transitive.

Now, we prove the denseness of periodic points. Now, without loss of generality, assume that c is even and let $Y = \{a/b \in \mathbb{Q} \cap X\}$, such b is odd and $\gcd(a, b) = 1$. So, this set becomes our set of periodic points. This is because, for any $y \in Y$, we cannot find any n such that the denominator becomes even. So, with this given denominator and range $[0, 1]$, there are only a finite number of values that $f^n(y)$ can take, and hence, it eventually circles back to itself, making it periodic. Now, to show that it is dense in X, observe that for any $x \in X$, we can find a subset J of X such that both endpoints belong to Y with the same denominator. Let $\epsilon > 0$ be arbitrary, so, by dividing J into halves of intervals containing x, by density theorem, we always find rational numbers in each iteration. Successively, we reach a point where we get an endpoint $y \in J$ such that $d(x, y) < \epsilon$. Now, if $y = a/2^k$ for some $k \geqslant 2$; we have that $z = a/(2^k + 1) < y$ and hence $d(z, x) < d(z, y) < \epsilon$, hence, we have found a point z in Y in neighborhood of x such that $d(z, x) < \epsilon$. Thus, the periodic points of f are dense in X. Since X is an infinite metric, it follows that f is sensitive as well and hence, f is Devaney chaotic for all $c \in \mathbb{N}; c > 1$.

On the other hand, observe that the Lyapunov constant is:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 |f'(x_n)| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 c = \log_2 c$$

Now, since we have already established that f is topologically transitive, the Lyapunov constant is positive when $c > 1$ and negative when $c < 1$, hence, f is Lyapunov chaotic when $c > 1$. Hence proved. □

Example 4.3.2. Let $X = [0, 1]$ be a metric space with standard topology and usual metric, then the map $f : X \to X$ defined by $f(x) = \mu x(1 - x); \mu \in (0, 4]$ is called the *Logistic Map.* We will prove that this is chaotic for $\mu = 4$.

Proof. Now observe that; for any μ; the map f achieves a maximum value at $x = 1/2$. Hence, for $\mu < 4$; the map achieves a maximum value of $\mu/4$ at $x = 1/2$. So, let A and B be two non-empty open subsets in X such that $A \subseteq [0, \mu/4)$ and $B \subseteq (\mu/4, 1])$. Now, observe that, for any amount of iterations on A, the maximum it can reach is $\mu/4$ and there does not exist any $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \varnothing$, hence, the logistic map is not transitive for $\mu < 4$.

Now, for $\mu = 4$, define a homomorphism from the *tent map*, denoted by g to the logistic map f given by :

$$h(x) = [\sin((\pi x)/2)]^2$$

Clearly, for $x \in [0,1]$, this map is surjective and by *definition 2.23*, it follows that since g is transitive, so is f, and hence, the logistic map is topologically transitive. Now, using *Theorem 4.2*, since X is an interval and we have proved above that is topologically transitive, it follows that f is chaotic. Hence proved. □

Example 4.3.3. Let $\mathbb{U} = [0,1)$ with usual metric and standard topology, and let $f : \mathbb{U} \to \mathbb{U}$ be a map defined by

$$f(x) = \begin{cases} 2x \text{ if } 2x \in \mathbb{U} \\ 2x - 1 \text{ otherwise} \end{cases} \tag{4.3.1}$$

This is called the *angle-doubler map.* We will prove that this map is chaotic.

Proof. Now, to prove transitivity, observe that from *Example 5.1*, when $c = 2$, for any two nonempty open subsets A and B of $\mathbb{U}$, we can find points of form $k_1/2^m$ for some $k_1, m \in \mathbb{N}$. We also showed that in successive iterations with such points, we eventually get the $\mathrm{n}^{\mathrm{the}}$ iteration as 1, which is 0 under this map. So, for
$x = k_1/2^m \in A$, $f^m(x) = 0$.
And since f is continuous, $f^m(A)$ includes an interval of form $[0, b)$ for some $b \in X$. Any interval of this form will have an image $\mathbb{U}$ after a finite number of iterations, say l. Hence, $n = m + l$ implies $f^n(A) = A$ and thus, $f^n(A) \cap B \neq \varnothing$. Hence proved that f is topologically transitive.
Now, for sensitivity, let $x \in \mathbb{U}$ and $U \in \mathcal{N}(x)$. Pick any $y \in U \backslash x$; then $\exists m \in \mathbb{N}$ such that $1/2^m \leqslant d(x,y) \leqslant 1/2^{m-1}$. Now, f doubles the distance between any 2 points with a gap less than 1/4. Hence, if $n - m = 2$, then $1/4 \leqslant d(f^n(x), f^n(y)) < 1/2$. Thus angle-doubler map is sensitive with a sensitivity level 1/4.

Finally, to show DPP, consider the set

$$D = \{a/b \in (\mathbb{Q} \cap X) | b \text{ is odd and } \gcd(a,b) = 1\}$$

. Clearly, for each b, there are only finite values the function can take, and hence, the points are periodic. However, unlike a tent map, there may be more than one orbit for any b. So, this set is dense in $\mathbb{U}$, as already shown in *Example 5.1.* Thus, the angle-doubler map has DPP and is chaotic.

Thus, our proof is complete. □

Example 4.3.4 (Shift Map)**.** Consider the *shift map* in *Definition 1.3.12*, we will prove that this map is chaotic.

Proof.

- The denseness of periodic points:

  Let a point be denoted by $x = (x_0 x_1 x_2 \ldots) \in \Sigma$ and let $\epsilon > 0$ be arbitrary. Now, for the set of periodic points to be dense in Σ we need to construct a periodic point within ϵ-neighborhood of x. So, we need a periodic point whose first $n+1$ terms agree with x.This is because for any $z = (z_0 z_1 z_2 \ldots) \in \Sigma$ with $z_k = x_k; k = 0, 1, \ldots, n$, by *Proximity Theorem, Definition 1.3.11*, $d(z,x) < 1/2^n$.And by Archimedean principle, we can find an $n \in \mathbb{N}$ with $1/2^n < \epsilon$.

  We take a point $y = (\overline{x_0 x_1 x_2 \ldots x_n})$. Clearly, from the above arguments,

$$d(x,y) < 1/2^n < \epsilon$$

  and y is periodic as well. Hence proved that periodic points of σ are dense in Σ.

- Transitivity: For this, choose any $x, y \in \Sigma$ and an $\epsilon > 0$. We will construct a z whose orbit coincides with orbits of both x and y. Now, choose an n such that $1/2^n < \epsilon$. From the above, for any new point having first $n+1$ terms same as x, it is within x and the same can be said for y.

Now, construct:

$$z = (\underbrace{0\ 1}_{\text{1 block}} \quad \underbrace{00\ 01\ 10\ 11}_{\text{2 blocks}} \quad \underbrace{000\ 001\ \ldots}_{\text{3 blocks}} \ \ldots)$$

This has been made by sticking together all sequences of length 1, then length 2, and so on. Thus, any sequence of length n will appear in z at any point in time. Now, for the $n+1$ terms of x, these are a sequence of length $n+1$, and hence by construction, they must be somewhere in z. So, there exists a k_1 such that the first $n+1$ terms of $\sigma^{k_1}(z)$ are the same as those of x,so, orbit of z coincides within ϵ of x.

Similarly, there exists a k_2 such that the first $n+1$ terms of $\sigma^{k_2}(z)$ are the same as that of y, so, the orbit of z coincides with y as well. Hence, we can say that after some iterations, orbits of x and y observe an intersection (inside z), hence, the shift map is transitive.

- Sensitivity: To establish a sensitivity, we choose a sensitivity level, say 1. Now, pick any $x \in \Sigma$ and any $\epsilon > 0$. Then pick any n such that $1/2^n < \epsilon$. Now, pick a point within ϵ of x. We will show that its orbit will diverge from that of x by at least 1.

  Let $y \neq x$ be such that $d(x, y) < 1/2^n$. So, this means that the first $n+1$ terms of x and y must be the same. And since $x \neq y$ there exists a $k > n$ such that $x_k \neq y_k$. So, considering these points, $\sigma^k(x) = x_k$ and $sigma^k(y) = y_k$,distance between them is:

$$d(x_k, y_k) = \sum_{n \geqslant 0} \frac{|s_{n+k} - t_{n+k}|}{2^n} \geqslant \frac{|s_k - t_k|}{2^0} = 1$$

  Hence proved that both points diverge by at least 1.

Hence, we conclude that the shift map is a chaotic dynamic system. □

Example 4.3.5 ([53]). Consider 2 cascades (X, f) and (Y, g) where $X = [0, 1]$ with the subspace topology of real line $\mathbb{R}$, $Y = [a, b]$ with discrete topology and continuous maps f, g on X, y, respectively, defined as:

$$f(x) = \begin{cases} 2x, & \text{for } x \in [0, 1/2] \\ 2 - 2x, & \text{for } x \in [1/2, 1] \end{cases}$$

$$g(a) = b \text{ and } g(b) = a.$$

Now, since the tent map is locally eventually onto, i.e., for every nonempty open subset $U \subseteq [0, 1]$, there exists a $k \in \mathbb{N}$ such that $f^n(U_=[0, 1]$ for all $n \geqslant k$. Also, the cascade (X, f) is Devaney chaotic. We claim that the product cascade $(X \times Y, f \times g)$ is strongly Auslander-Yorke as well as strongly Ruelle-Takens chaotic.
Let $U_1 \times V_1, U_2 \times V_2$ be any pair of nonempty open subsets of $X \times Y$. Using locally eventually ontoness and transitivity of f, we can find an $n \in \mathbb{N}$ such that $[(f \times g)^m(U_1 \times V_1)] \cap (U_2 \times V_2) \neq \varnothing$. Hence, the product is topologically transitive. Note that $X \times Y$ is a separable, complete metric space therefore the property of semiflows that the product being transitive implies that both semiflows are transitive, and transitivity of $(X \times Y, f \times g)$ implies point transitivity. Let $x \in X$ be a periodic point of period m and y be a point of Y. Also, consider $m\mathbb{N} \cap 2\mathbb{N} = k\mathbb{N}$ and consider the sequence t_n, where $t_n = kn$ for each $n \in \mathbb{N}$. Clearly, sequence t_n diverges to infinity and $t_n(x, y) \to (x, y)$. So, this, together with the fact that the set of periodic points of f is dense in X implies that $(X \times Y, f \times g)$ has a dense set of recurring points. Now, we also know that the product of 2 semiflows is sensitive if and only if at least one of the factors is sensitive, so, $(X \times Y, f \times g)$ is sensitive. Thus, we have proved that (X, f) is strongly Auslander-Yorke and Ruelle-Takens chaotic but since (Y, g) is not sensitive, it is not chaotic in any of these terms.
Now, let $Y = y$ be a discrete space and $g : Y \to Y$ be the identity map. Now, consider the product cascade (X_∞, ϕ_∞) where $(X_1, \phi_1) = (X, f)$ and $(X_i, \phi_i) = (Y, g)$

for rest of the $i's$. We can see that the cascade (X_∞, ϕ_∞) is both strongly Auslander-Yorke and strongly Ruelle-Takens chaotic. But for any $i \geqslant 2$, (X_i, ϕ_i) is not sensitive and hence not chaotic in either sense.

Example 4.3.6. Consider cascades (X, f) and (Y, g) where $X = (0, \infty)$ with subspace topology of real line $\mathbb{R}$.
$Y = \Sigma_2 = \{(s_0 s_1 s_2 \ldots) : s_j = 0 \text{ or } 1 \text{ for each } j \in \mathbb{N}_0\}$, the map $f : X \to X$ is defined by $f(x) = 2x$ and the map $g : Y \to Y$ is the left shift map, also, we have that:

$$s^\star = (\underbrace{\ 0\ 1\ }_{1\text{ block}}\quad \underbrace{00\ 01\ 10\ 11}_{2\text{ blocks}}\quad \underbrace{000\ 001\ \ldots}_{3\text{ blocks}}\ \ldots)$$

is an unpredictable point of the cascade (Σ_2, f_2). In fact, it is a transitive point as well. Hence, it is Poincaré chaotic. However, the cascade $(X \times Y, f \times g)$is not Poincaré chaotic since if form some sequence t_n diverging to infinity,
$t_n(x, y) \to (x, y)$, then it will imply that $2^{t_n}x \to x$ which is not possible.

Example 4.3.7 ([53]). Consider the cascade (X, f) where $X = [0, 2]$ and $f : X \to X$ is the continuous map given by:

$$f(x) = \begin{cases} 2x + 1, & \text{for } x \in [0, 1/2] \\ -2x + 3, & \text{for } x \in [1/2, 1] \\ -x + 2, & \text{for } x \in [1, 2] \end{cases}$$

Observe that in this, (X, f) is clearly Devaney chaotic. Every periodic point is not a recurrent point and the set of all recurrent points of (X, f) is dense in X. Moreover, X being a compact separable metric space, we can say that topological transitivity implies point transitivity. Hence, (X, f) is strongly Auslander-Yorke and Strongly Ruell-Takens chaotic. However, the extension of this to the factor map $(X \times X, f \times f)$ has no topological and hence point transitivity, hence, it is neither strongly Auslander-Yorke nor strongly Ruelle-Takens chaotic.

Conclusion

In this chapter we studied various types of chaos which include Devaney's Chaos, Li-Yorke Chaos, Wiggins Chaos and more. **Theorem 4.1.2** helped us prove that Devaney's Chaos is a stronger form of chaos compared to Wiggins Chaos and Martelli Chaos. We studied various examples from concrete spaces like Tent Map in **Example 4.3.1** and Shift Map in **Example 4.3.4** which helped us further our understanding about Chaos.In **Example 4.3.6** and **Example 4.3.7** we analysed Cascades which displayed stronger forms of Chaos like Poincaré Chaos and Auslander-Yorke Chaos.

Chapter 5

On Product of Dynamical Systems

The product of Dynamical Systems refers to maps on the Cartesian product spaces. In this chapter, we mainly discuss how the chaotic conditions on dynamical systems carry over to their products. We also analyse the properties that are satisfied by these dynamical systems. Then we see what sub-conditions we need to take care of for the product to satisfy those properties. We demonstrate that if two cascades(or even one of them)are sensitive, their product is also sensitive. Additionally, we give several sufficient conditions under which the product of two chaotic cascades(in the sense of Devaney) are chaotic. Then we discuss finite and infinite product systems. Li-Yorke Chaos of the product dynamical system $(\prod X_i, \prod f_i)$ has been studied when each of the factor dynamical systems (X_i, f_i) is Li-Yorke chaotic and vice-versa.

5.1 Transitivity

In this section, we show how transitive property of product maps affects the transitive property of each dynamical system and vice versa.

Theorem 5.1.1 ([47]). Let (X, f) and (Y, g) be cascades. Then The following hold:

1. 1. If the product map $f \times g$ is topologically transitive, then f and g are topologically transitive.

2. 2. If the product map $f \times g$ is topologically weakly mixing, then f and g are topologically weakly mixing

Proof. Let $U_1, U_2 \subset$ X and $V_1, V_2 \subset$ Y be non-empty open sets. Then the sets $U = U_1 \times Y$, $V = U_2 \times$ Y ,$P = X \times V_1$ and $Q = X \times V_2$ are open in $X \times Y$. Clearly, $N_{f\times g}(U, V) = N_f(U_1, U_2)$ and $N_{f\times g}(\mathrm{P}, \mathrm{Q}) = N_g(V_1, V_2)$. Therefore, by the definitions, We may construct the product map $(X \times Y, h)$ by defining $h^n(x, y) = (h^n x, h^n y)$. We call (X, h) and (Y, h) the factors of the product map and we know that the proof will hold. However, the converse of this theorem is not true. □

Theorem 5.1.2 ([48]). Let $f : X \to X$ and $g : Y \to Y$ be two cascades. If g is topologically mixing, then the following hold.

1. If f is topologically transitive, then so does the product of the dynamical systems $f \times g$.

2. If f is syndetically transitive, then so does the product of the dynamical systems $f \times g$.

Proof. For any nonempty open sets $W_1, W_2 \subset X \times Y$, there exist nonempty open sets $U_1, U_2 \subset X$and $V_1, V_2 \subset Y$ with $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$. Obviously, $N_{f\times g}(U_1 \times V_1, U_2 \times V_2) = N_f(U_1, U_2) \cap N_g(V_1, V_2)$.
As g is topologically mixing, there is $M > 0$ such that
$N_g(V_1, V_2) \supset [M, +\infty)$. Since f is continuous, $f^{-M}(U_2)$ is a nonempty and open subset of X.

1. If f is topologically transitive, $N_f(U_1, f^{-M}(U_2)) \neq \emptyset$, which implies that $N_f(U_1, U_2) \cap N_g(V_1, V_2) \neq \emptyset$. Consequently, the product map $f \times g$ is topologically transitive.

2. If f is syndetically transitive, by hypothesis and the definition, there exists $n \in \mathbb{N}$ such that

$N_f(U_1, f^{-M}(U_2)) \cap [m, m+n] \neq \varnothing$ for every $m \in \mathbb{N}$, which implies that

$N_f(U_1, U_2) \cap N_g(V_1,)V_2 \cap [m, m+n+M] \neq \varnothing$.

Consequently, the product map $f \times g$ is syndetically transitive. $\square$

Lemma 13. *The product of two topologically mixing maps is topologically mixing.*

Proof. Let $f : X \to X$ and $g : Y \to Y$ be topologically mixing maps. Given $W_1, W_1 \subset X \times Y$, there exists open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$. such that $U_1 \times V_2 \subset W_1$ and $U_2 \times V_2 \subset W_2$. By assumption there exist n_1 and n_2 such that $f^n(U_1) \cap U_2 \neq \varnothing$ for $n \geqslant n_1$ and $g^n(V_1) \cap V_2 \neq \varnothing$ for $n \geqslant n_2$. For n $\geqslant n_0 = \max n_1, n_2$,we get

$$
\begin{aligned}
cl[(f \times g)^n(U_1 \times V_1)] \cap (U_2 \times V_2) &= [f^n(U_1) \times g^n(V_1)] \cap (U_2 \times V_2) && (5.1.1)\\
&= [f^n(U_1) \cap U_2][g^n(V_1) \cap V_2] \neq \varnothing && (5.1.2)
\end{aligned}
$$

which means that $f \times g$ is topologically mixing. $\square$

Theorem 5.1.3. If f_∞^* (resp., f_N^*) is transitive, then each factor map f_i is transitive. The converse is not true.

Proof. Given any fixed positive integer i, for any pair of non-empty open subsets A_i, B_i of X_i, as f_∞^* is transitive, then there exists some positive integer n such that

$$\left((f_\infty^*)^n(A_i \times \prod_{j \neq i} X_j)\right) \cap \left(B_i \times \prod_{j \neq i} X_j\right) = (f_i^n(A_i) \cap B_i) \times \prod_{j \neq i} f_j^n(X_j) \neq \varnothing.$$

This implies that $f_i^n(A_i) \cap B_i \neq \varnothing$, so f_i is transitive. $\square$

5.2 Periodic points of the product systems

First, we consider the finite product systems.

Proposition 6 ([46]). *For each integer $N \geqslant 2$,*

$$Per(f_N^*) = \prod_{i=1}^{N} Per(f_i).$$

Theorem 5.2.1 ([46]). If $Per(f_i) \neq \varnothing$ holds for each positive integer i, then $\overline{Per(f_\infty^*)} = \overline{\prod_{i=1}^{\infty} Per(f_i)}$ if and only if $\sup\{\min P(f_i) : i \in \mathbb{N}\} < +\infty$.

Proof. ($\Rightarrow$) It is obvious that $\mathrm{Per}(f_\infty^*) \neq \varnothing$ holds since $\overline{Per(f_\infty^*)} = \overline{\prod_{i=1}^{\infty} Per(f_i)} = \overline{\prod_{i=1}^{\infty} Per(f_i)} \neq \varnothing$. For any $\{p_i\} \in Per(f_\infty^*)$, we have that there exists some positive integer m such that $(f_\infty^*)^m(\{p_i\}) = \{f_i^m(p_i)\} = \{p_i\}$, which implies $f_i^m(p_i) = p_i$ holds for each $i \in \mathbb{N}$. So $\sup\{\min P(f_i) : i \in \mathbb{N}\} \leqslant m < +\infty$.
($\Leftarrow$) First, choosing arbitrarily $x = \{x_i\} \in \overline{\prod_{i=1}^{\infty}}\mathrm{Per}(f_i)$, it is easy to see that for any $A = \prod_{i=1}^{\infty} A_i \in \mathbb{N}(x)$ satisfying each $A_i \in \mathbb{N}(x_i)$ and $\{i \in \mathbb{N} : A_i \neq X_i\}$ is finite, $(\prod_{i=1}^{\infty} \mathrm{Per}(f_i)) \cap A \neq \varnothing$. Take $\{p_i\} \in (\prod_{i=1}^{\infty} \mathrm{Per}(f_i)) \cap A$ and put $p_i^* \in \mathrm{Per}(f_i)$ such that $f_i^{\min P(f_i)}(p_i^*) = p_i^*$ for each positive integer i. Let $j = \max\{i \in \mathbb{N} : A_i \neq X_i\}$, then P $:= (p_1, \ldots, p_j, p_{j+1}^*, p_{j+2}^*, \ldots) \in (\prod_{i=1}^{\infty} \mathrm{Per}(f_i)) \cap$ A.
Let us choose $k_0 = \max\{m_i : 1 \leqslant i \leqslant j\} + \sup\{\min P(f_i) : i \in \mathbb{N}\}$, where m_i is the period of the point p_i under f_i. Then,

$$(f_\infty^*)^{k_0}(P) = (f_1^{k_0}(p_1), \ldots, f_j^{k_0}(p_j), f_{j+1}^{k_0}(p_{j+1}^*), \ldots) = P \in \mathrm{Per}(f_\infty^*) \cap A.$$

Thus, $\overline{\prod_{i=1}^{\infty} \mathrm{Per}(f_i)} \subseteq \overline{\mathrm{Per}(f_\infty^*)}$.
On the other side, it is clear that $\overline{\mathrm{Per}(f_\infty^*)} \subseteq \overline{\prod_{i=1}^{\infty} \mathrm{Per}(f_i)}$.
Hence, $\overline{\mathrm{Per}(f_\infty^*)} = \overline{\prod_{i=1}^{\infty} \mathrm{Per}(f_i)}$. □

5.3 Sensitivity and Chaos

In this section, we discuss how different types of sensitivities in dynamical systems results in the sensitivity of their finite product maps and also give some conditions for their infinite product maps to be sensitive. Then we talk about some subconditions that need to be fulfilled for a dynamical system to be Devaney chaotic and later in this section, we also talk about some other types of chaos.

Theorem 5.3.1 ([47]). Let X and Y be metric spaces with metrics d_x and d_y, respectively, and let $f : X \to X$ and $g : Y \to Y$ be not-necessarily continuous maps. Then the following hold:

1. If f or g is syndetically sensitive, then $f \times g$ is syndetically sensitive.
2. If f or g is cofinitely sensitive, then $f \times g$ is cofinitely sensitive.
3. If f or g is multi-sensitive, then $f \times g$ is multi-sensitive.
4. If $f \times g$ is ergodically sensitive if and only if f or g is ergodically sensitive.

Proof. Let $U \subset X$ and $V \subset Y$ be nonempty open sets. Then, for any $\delta > 0$, one can easily verify that $N_{f\times g}(U \times V, \delta) \supset N_f(U, \delta)N_g(V, \delta)$. Therefore, parts 1), 2), and 3) of the theorem are true.

From the above argument, it is easy to see that if f or g is ergodically sensitive, then so is the product map $f \times g$.

Now we suppose that the product map $f \times g$ is ergodically sensitive and that both f and g are not ergodically sensitive. This means that for any given $\delta > 0$, there exists a certain open set $U \subset X$ with $d(N_f(U, \delta/3)) = 0$. Similarly, there exists a certain open set $V \subset Y$ with $d(N_g(V, \delta/3)) = 0$. It is easy to see that $N_{f\times g}(\mathrm{U} \times \mathrm{V}, \delta)$ $\subset N_f(U, 1/3\delta)Ng(V, 1/3\delta)$. This implies that

$d(N_{f\times g}(U \times V, \delta)) \leqslant d(N_f(U, 1/3\delta)Ng(V, 1/3\delta)) \leqslant d(N_f(U, 1/3) + d(N_g(V, 1/3\delta)) = 0.$

It is a contradiction.

So, the proof of part 4) is completed.

Thus, the entire proofs are ended. □

Theorem 5.3.2 ([56]). f_∞ is sensitive if and only if there exists a positive integer k such that f_k is sensitive.

Proof. Let f_∞ be sensitive with constant c. We have to show that there exists a positive integer k such that f_k is sensitive. Suppose to the contrary, i.e., that each f_i is non-sensitive. Then for each i there exists a nonempty open subset U_i of X_i such that $d_i(f_i^n(x), f_i^n(y)) < \frac{c}{2}$, for all $x, y \in U_i$ and $n \in \mathbb{N}$. Let us take a fixed positive integer n_0 such that $\sum_{n=n_0}^{\infty} \frac{1}{2^n} < \frac{c}{2^2}$. Note that $A = \prod_{i=1}^{n_0} U_i \times \prod_{i=n_0+1}^{\infty} X_i$ is a nonempty open subset of X_∞. Now,for any pair of elements x_i, $y_i \in A$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(f_\infty^n(x_i), f_\infty^n((y_i))) &= \sum_{i=1}^{n_0} \frac{1}{2^i} d_i(f_i^n(x_i), f_i^n(y_i)) \\ &< \sum_{i=1}^{n_0} \frac{1}{2^i} \left(1 + d_i(f_i^n(x_i), f_i^n(y_i))\right) \\ &< \sum_{i=1}^{n_0} \frac{1}{2^i} \frac{c}{2} \\ &< \frac{c}{2} \left(1 + \sum_{i=n_0+1}^{\infty} \frac{1}{2^i}\right) \\ &< c, \end{aligned}$$

a contradiction to our hypothesis. Hence, there exists a positive integer k such that f_k is sensitive.

Conversely, let f_k be sensitive with sensitivity constant c for some fixed positive integer k. Let $x = (x_i) \in X_\infty$ and A be an open neighborhood of (x_i). Note that $p_k(A)$ is an open neighborhood of x_k in X_k, where $p_k : \prod_{i=1}^{\infty} X_i \to X_k$ is the kth projection map. Now, since f_k is sensitive, there exists a point $y \in p_k(A)$ and $n_0 \in \mathbb{N}$ such that $d_k(f_k^{n_0}(x_k), f_k^{n_0}(y)) > c$. Let us choose $z = (z_i) \in X_\infty$, where $z_i = x_i$ when $i \neq k$ and $z_k = y$. Clearly, z is an element of A and

$$\begin{aligned} &d(f_\infty^{n_0}(x), f_\infty^{n_0}(z)) \\ &= \frac{1}{2^k} d_k(f_k^{n_0}(x_k), f_k^{n_0}(y)) \\ &> \frac{1}{2^k} \left(1 + d_k(f_k^{n_0}(x_k), f_k^{n_0}(y))\right) \\ &> \frac{1}{2^k} \frac{c}{1+c}. \end{aligned}$$

Hence, φ_∞ is sensitive with constant $\frac{1}{2^k}\frac{c}{1+c}$. $\square$

Theorem 5.3.3 ([51]). (X_∞, f_∞) is multisensitive if and only if there exists a positive integer k such that (X_k, f_k) is multisensitive.

Proof. Let (X_∞, f_∞) be multisensitive with constant c. Suppose for each i,

$$(X, f_i) \text{ is not multisensitive.}$$

Let us take a fixed positive integer n_0 such that

$$\sum_{i=n_0+1}^{\infty} \frac{1}{2^i} < \frac{c}{2}.$$

Since each factor-map is not multisensitive, corresponding to each (X, f_i), $1 \leqslant i \leqslant n_0$, there exist nonempty open subsets $U_{1,i}, U_{2,i}, \ldots, U_{k_i,i}$ of X_i such that

$$\bigcap_{j=1}^{k_i} D(U_{j,i}, \frac{c}{2}) = \varnothing,$$

where $k_1, k_2, \ldots, k_{n_0}$ are fixed positive integers. Now, consider open subsets,

$$A(j_1, j_2, \ldots, j_{n_0}) = U_{j_1,1} \times U_{j_2,2} \times \cdots \times U_{j_{n_0},n_0} \times \prod_{i=n_0+1}^{\infty} X_i,$$

where $1 \leqslant j_i \leqslant k_i$ and $1 \leqslant i \leqslant n_0$ of X_∞. By multisensitivity of (X_∞, f_∞), we get a $k_0 \in \mathbb{N}$ such that

$$\operatorname{diam}(f_\infty^{k_0}(A(j_1, j_2, \ldots, j_{n_0}))) > c,$$

for $1 \leqslant j_i \leqslant k_i$ and $1 \leqslant i \leqslant n_0$.

Now, as $\bigcap_{j=1}^{k_i} D(U_{j,i}, \frac{c}{2}) = \varnothing$ for each $1 \leqslant i \leqslant n_0$, corresponding to k_0 there exists an open subset $U^i_{k_0,i}$ of X_i, where $k_{0i} \in \{1, 2, \ldots, k_i\}$ such that $\operatorname{diam}(f_i^{k_0}(U^i_{k_0,i})) < \frac{c}{2}$, for $1 \leqslant i \leqslant n_0$. Consider the nonempty open subset

$$B = U_{k_{01},1} \times U_{k_{02},2} \times \cdots \times U_{k_{0n_0},n_0} \times \prod_{i=n_0+1}^{\infty} X_i$$

of X_∞. Then for any $(x_i), (y_i) \in B$ we have

$$\begin{aligned}
& d(f_\infty^{k_0}(x_i), f_\infty^{k_0}(y_i)) \\
& < \sum_{i=1}^{n_0} \frac{1}{2^i} d_i(f_i^{k_0}(x_i), f_i^{k_0}(y_i)) \\
& < \sum_{i=1}^{n_0} \frac{1}{2^i} \left(1 + d_i(f_i^{k_0}(x_i), f_i^{k_0}(y_i))\right) \\
& < \frac{c}{2} \left(1 + \frac{c}{2} + \sum_{i=n_0+1}^{\infty} \frac{1}{2^i}\right) \\
& < c,
\end{aligned}$$

a contradiction to the earlier inequality. Hence, there must exist a positive integer k such that (X_k, f_k) is multisensitive.

Conversely, let (X_k, f_k) be multisensitive with constant c for some fixed positive integer k. Again, let $A_1, A_2, \ldots, A_n$ be finitely many nonempty open subsets of X_∞. Note that for $1 \leqslant i \leqslant n$, $p_k(A_i)$ is a nonempty open subset of X_k, where $p_k : X_\infty \to X_k$ is the kth projection map. Since (X_k, f_k) is multisensitive, there exists a $k_0 \in \mathbb{N}$ which lies in $\bigcap_{i=1}^n D(p_k(A_i), c)$. Now we claim that $k_0 \in D(A_i, \frac{c}{2k(1+c)})$, for $1 \leqslant i \leqslant n$. Since for each $1 \leqslant i \leqslant n$, $k_0 \in D(p_k(A_i), c)$, there exist $x_{i,k}$ and $y_{i,k}$ in $p_k(A_i)$ such that $d_k(f_k^{k_0}(x_{i,k}), f_k^{k_0}(y_{i,k})) > c$. For any fixed $1 \leqslant i \leqslant n$, as A_i is nonempty let (x_j) be an element of A_i. Taking two points (y_j) and (z_j) of A_i, where $y_j = z_j = x_j$ when $j \neq k$, $y_k = x_{i,k}$, and $z_k = y_{i,k}$, we get

$$\begin{aligned}
& d(f_\infty^{k_0}(y_j), f_\infty^{k_0}(z_j)) \\
& = \frac{1}{2k} d_k(f_k^{k_0}(x_{i,k}), f_k^{k_0}(y_{i,k})) \\
& < \frac{1}{2k} \left(1 + d_k(f_k^{k_0}(x_{i,k}), f_k^{k_0}(y_{i,k}))\right) \\
& > \frac{1}{2k} \frac{c}{1+c}.
\end{aligned}$$

Therefore, $t_0 \in \bigcap_{i=1}^n D(A_i, \frac{c}{2k(1+c)})$. Hence, (X_∞, f_∞) is multisensitive with constant $\frac{c}{2k(1+c)}$. $\square$

Theorem 5.3.4 ([53]). (X_∞, f_∞) is ergodically sensitive if and only if there exists a positive integer k such that (X_k, f_k) is ergodically sensitive.

Proof. Let (X_∞, f_∞) be ergodically sensitive with constant c. Suppose for each $i \neq k$, (X, f_i) is not ergodically sensitive. Choose a fixed positive integer n such that $\sum_{i=n}^{\infty} \frac{1}{2^i} < \frac{c}{2}$. Since each factor-map is not ergodically sensitive, corresponding to each (X, f_i), $1 \leqslant i \leqslant n$, there exists a nonempty open subset U_i of X_i such that $\overline{d}D(U_i, \frac{c}{2}) = 0$. As $A = \prod_{i=1}^{n} U_i \times \prod_{i=n+1}^{\infty} X_i$ is a nonempty open subset of X_∞ and (X_∞, f_∞) is ergodically sensitive, $dD(A, c) > 0$. Now we show that

$$D\left(\prod_{i=1}^{n} U_i \times \prod_{i=n+1}^{\infty} X_i, c\right) \subseteq \bigcap_{i=1}^{n} D\left(U_i, \frac{c}{2}\right).$$

Let $n_0 \in D(A, c)$, then there exist $(x_i), (y_i) \in A$ such that $d(f_\infty^{n_0}(x_i), f_\infty^{n_0}(y_i)) > c$, i.e.,

$$\sum_{i=1}^{n} \frac{1}{2^i} d_i(f_i^{n_0}(x_i), f_i^{n_0}(y_i)) + \sum_{i=n+1}^{\infty} \frac{1}{2^i} > c.$$

Since $\sum_{i=n}^{\infty} \frac{1}{2^i} < \frac{c}{2}$, $\sum_{i=1}^{n} \frac{1}{2^i} d_i(f_i^{n_0}(, x_i), f_i^{n_0}(, y_i)) + \sum_{i=n+1}^{\infty} \frac{1}{2^i} > \frac{c}{2}$, which implies there exists a $j \in \{1, 2, \ldots, n\}$ such that $d_j(f_j^{n_0}(x_j), f_j^{n_0}(y_j)) > \frac{c}{2}$, for if $d_i(f_i^{n_0}(x_i), f_i^{n_0}(y_i)) \leqslant \frac{c}{2}$ for each $i \in \{1, 2, \ldots, n\}$, then

$$\sum_{i=1}^{n} \frac{1}{2^i} d_i(f_i^{n_0}(x_i), f_i^{n_0}(y_i)) + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \leqslant \frac{c}{2} + \frac{c}{2} < c,$$

a contradiction. Therefore, $n \in D(U_j, \frac{c}{2})$ and (6) holds true. Consequently,

$$\overline{d}D(A, c) \leqslant \overline{d}D\left(\bigcap_{i=1}^{n} D\left(U_i, \frac{c}{2}\right)\right) = 0$$

which is a contradiction. Hence, there exists a positive integer k such that (X_k, f_k) is ergodically sensitive.

Conversely, let (X_k, f_k) be ergodically sensitive with constant c for some fixed positive integer k. Let A be a nonempty open subset of X_∞. Then $p_k(A)$ is a nonempty open subset of X_k. Ergodic sensitivity of (X_k, f_k) implies that $\overline{d}D(p_k(A), c) > 0$. Now, we show that

$$D(p_k(A), c) \subseteq D\left(A, \frac{c}{2k(1+c)}\right).$$

Let n_0 be an element of $D(p_k(A), c)$. Then $d_k(f_k^{n_0}(y), f_k^{n_0}(z)) > c$ for some $y, z \in p_k(A)$. As A is nonempty, there exists (x_i) in A. Taking two points (y_i) and (z_i) of A, where $y_i = z_i = x_i$ when $i \neq k$, $y_k = y$ and $z_k = z$, we get that

$$d(f_\infty^{n_0}(y_i), f_\infty^{n_0}(z_i)) > \frac{1}{2k}\frac{c}{1+c}.$$

Therefore, (7) holds true. Further,

$$\overline{d}D\left(A, \frac{c}{2k(1+c)}\right) \geqslant \overline{d}D\left(D(p_k(A), c)\right) > 0.$$

Hence, (X_∞, f_∞) is ergodically sensitive with constant $\frac{c}{2k(1+c)}$. □

In the following example, we show that (X_∞, f_∞) is sensitive, multisensitive, and ergodically sensitive, but no (X_i, f_i) except (X_1, f_1) is sensitive, multisensitive, and ergodically sensitive.

Example 5.3.1. Consider the cascade (X_∞, f_∞), where $X_1 = [0, \infty)$ and $X_i = S^1$ for all $i \neq 1$ with usual topologies. We are taking the map f_∞ to be defined by $f_\infty^n(x_i) = (2nx_1, e^{2\pi i n}x_2, e^{2\pi i n}x_3, \ldots)$. Let $c > 0$ be a fixed real number. Further, let U be a nonempty open subset of X_1. Taking any two distinct points $p, q \in U$, we can certainly find a positive natural number n_0 such that $2k|p-q| > c$, for all $k \in [n_0, \infty)$. Clearly, $\overline{d}D([n_0, \infty)) > 0$ so (X_1, f_1) is ergodically sensitive. Again, let k be any fixed natural number and $U_1, U_2, \ldots, U_k$ be nonempty open subsets of X_1, then there exists a positive natural number n_1 such that $\bigcap_{i=1}^k D(U_i, c) \supseteq [n_1, \infty)$. Thus (X_1, f_1) is multisensitive as well. we know that (X_∞, f_∞) is sensitive, multisensitive, and ergodically sensitive. Note that for all $i \neq 1$, $d_i(x_i, y_i) = d_i(f^n(x_i), f^n(y_i))$ for all

$x_i, y_i \in X_i$ and $n \in \mathbb{N}$. Hence, (X_i, f_i) is not sensitive for all $i \neq 1$

The following example shows that the product of two chaotic maps need not be chaotic.

Example 5.3.2. Let $f : [0, 2] \to [0, 2]$ be defined as follows:

$$f(x) = \begin{cases} 2x + 1 & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ -2x + 3 & \text{for } \frac{1}{2} \leqslant x \leqslant 1, \\ -x + 2 & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

Then, the map f is chaotic, but $f \times f : [0, 2] \times [0, 2] \to [0, 2] \times [0, 2]$ is not chaotic.

Sub-conditions of chaos

Lemma 14 ([56]). *Let X and Y be metric spaces with metrics d_1 and d_2 , respectively, $f : X \to X$ and $g : Y \to Y$ be not-necessarily continuous maps.*
i) If f or g is sensitively dependent on initial conditions, then $f \times g : X \times Y \to X \times Y$ is sensitively dependent on initial conditions.
ii) If $f \times g : X \times Y \to X \times Y$ is sensitively dependent on initial conditions, then at least one of f or g is sensitively dependent on initial conditions.

Proof. i) Let us assume f is sensitively dependent on initial conditions. Then we will show that the same is true for $f \times g$.
Let $p = (x, y) \in X \times Y$ be any point and W any neighborhood of p. Then there exist open neighborhoods U of $xinX$ and V of y in such that $U \times V \subset W$. As f is sensitively dependent on initial conditions, there exists $\epsilon > 0$ such that for a certain $x' \in U$ and an integer $n > 0$ the inequality $d_1(f^n(x), f^n(x')) > \epsilon$ holds.Then for any

$y' \in V, p' = (x', y')$ belongs to W and

$$\begin{aligned} d((f \times g)_n(p), (f \times g)_n(p')) &= d_1(f^n(x), f^n(x')) + +d_2(g^n(y), g^n(y')) \\ &\geqslant d_1(f^n(x), f^n(x')) > \epsilon. \end{aligned}$$

This means that $f \times g$ is sensitively dependent on initial conditions.

ii) Let us assume that both f and g are not sensitively dependent on initial conditions. This means that,given any $\epsilon > 0$ there exists $x \in X$ such that for a certain open set $U \subset X$ containing x, the inequality

$$d_1(f^n(x), f^n(x')) > \epsilon/2$$

holds for every $x' \in$ U and positive integer n. Similarly, there exists $y \in Y$ such that for a certain open set $V \subset Y$ containing y , the inequality

$$d_1(g^n(x), g^n(x')) > \epsilon/2$$

holds for every $y' \in V$ and positive integer n. Then we get

$$d((f \times g)_n(p), (f \times g)_n(p')) = d_1(f^n(x), f^n(x')) + +d_2(g^n(y), g^n(y')) < \epsilon$$

for $(x', y') \in U \times V$. This means that $f \times g$ is not sensitively dependent on initial conditions, contradicting the hypothesis. □

Lemma 15 ([56]). *Let X and Y be metric spaces with metrics d_1 and d_2 respectively, $f : X \to X$ and $g : Y \to Y$ (not-necessarily continuous) maps. The set of periodic points of $f \times g$ is dense in $X \times Y$ if and only if, for both of f and g the sets of periodic points in X and Y are dense (in X, resp. Y)*

Proof. Let us assume that the set of periodic points of f is dense in X and the set of periodic points of g is dense in Y. Let us see that the set of periodic points of $f \times g$ is dense in $X \times Y$. Let $W \subset X \times Y$ be any non-empty open set. Then there exist non-empty open sets $U \subset X$ and $V \subset Y$ with $U \times V \subset W$. By assumption, there exists a point $x \in U$ such that $f^n(x) = x$ with $n > 0$. Similarly, there exists $y \in V$ such that $g^m(y) = y$ with $m > 0$. For $p = (x, y) \subset W$ and $k = mn$ we get

$$(f \times g)^k(p) = (f \times g)^k(x, y) = (f^k(x), g^k(y)) = (x, y)$$

This means that W contains a periodic point and thus the set of periodic points of $f \times g$ is dense in $X \times Y$.

Conversely, let $U \subset X$ and $V \subset Y$ be non-empty open subsets. Then $U \times V$ is a non-empty open subset of $X \times Y$. As the set of the periodic points of $f \times g$ is dense in $X \times Y$, there exists a point $p = (x, y) \in U \times V$ such that $(f \times g)^n(x, y) = (f^n(x), g^n(y)) = (x, y)$ for some n. From the last equality we obtain $f^n(x) = x$ for $x \in U$ and $g^n(y) = y$ for $y \in Y$.

By Lemma 2 and Lemma 3, sensitive dependence on initial conditions and denseness of periodic points carry over from factors to products. □

Theorem 5.3.5 ([56]). Let $f : X \to X$ and $g : Y \to Y$ be not-necessarily continuous, chaotic, and topologically mixing maps on the metric spaces X and Y. Then $f \times g : X \times Y \to X \times Y$ is chaotic.

Proof. The map $f \times g$ is said to be chaotic, if these three conditions of Devaney are satisfied :

1. Sensitively dependent on initial conditions.

2. It has dense periodic points.

3. It is topologically mixing.

To show : The map $f \times g$ is sensitively dependent on initial conditions.

Let us assume f is sensitively dependent on initial conditions. Then we will show that the same is true for $f \times g$.

Let $p = (x, y) \in X \times Y$ be any point and W any neighborhood of p. Then there exist open neighborhoods U of $x' \in X$ and V of $y' \in Y$ such that $U \times V \subset W$. As f is sensitively dependent on initial conditions, there exists $\varepsilon > 0$ such that for a certain $x \in U$ and an integer $n > 0$ the inequality $d1(f_n(x), f_n(x')) > \varepsilon$ holds. Then for any $y' \in V$, $p' = (x', y')$ belongs to W and

$$d((f\times g)_n(p), (f\times g)_n(p')) = d1(f_n(x), f_n(x'))+d2(g_n(y), g_n(y')) \geqslant d1(f_n(x), f_n(x')) > \varepsilon$$

This means that $f \times g$ is sensitively dependent on initial conditions.

To show : The map $f \times g$ has dense periodic points.

Let us assume that the set of periodic points of f is dense in X and the set of periodic points of g is dense in Y. Let us see that the set of periodic points of $f \times g$ is dense in $X \times Y$. Let $W \subset X \times Y$ be any non-empty open set. Then there exist non-empty open sets $U \subset X$ and $V \subset Y$ with $U \times V \subset W$. By assumption, there exists a point $x \in U$ such that $f_n(x) = x$ with $n > 0$. Similarly, there exists $y \in V$ such that $g_m(y) = y$ with $m > 0$. For $p = (x, y) \in W$ and $k = mn$ we get,

$$(f \times g)_k(p) = (f \times g)_k(x, y) = (f_k(x), g_k(y)) = (x, y)$$

This means that W contains a periodic point and thus the set of periodic points of $f \times g$ is dense in $X \times Y$.

To Show : The map $f \times g$ is topologically mixing.

Let $f : X \to X$ and $g : Y \to Y$ be topologically mixing maps. Given $W_1, W_2 \subset X \times Y$, there exists open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$, such that, $U_1 \times V$ that $f_n(U_1) \cap U_2 = \varnothing$ for $n \geqslant n_1$ and $g_n(V_1) \cap V_2 = \varnothing$ for $n \geqslant n_2$. For $n \geqslant n_0 = max\{n_1, n_2\}$ we get,

$$[(f \times g)_n(U_1 \times V_1)] \cap (U_2 \times V_2) = [f_n(U_1) \times g_n(V_1)] \cap (U_2 \times V_2)$$

$$= [f_n(U_1) \cap U_2] \times [g_n(V_1) \times V_2] = \varnothing$$

which means that $f \times g$ is topologically mixing, hence topologically transitive. Thus all three conditions of Devaney chaos are satisfied. □

Theorem 5.3.6. Let X be a metric space, $f : X \to X$ a continuous and chaotic map; $g : Y \to Y$ a not-necessarily continuous, chaotic, and topologically mixing map on the metric space Y. Then $f \times g : X \times Y \to X \times Y$ is chaotic.

Proof. It is enough to show that $f \times g$ is topologically transitive. It is obviously enough to show this for open sets of the form $U \times V$. So, let be given two sets $U_1 \times V_1$ and $U_2 \times V_2$ with U_1, U_2 open in X and V_1, V_2 open in Y. As g is topologically mixing, there exists $n_0 > 0$ with $g_n(V_1) \cap V_2 \neq \phi$ for all $n \geqslant n_0$. On the other hand, there exists a periodic point $x \in U_1$ whose orbit enters U_2. Thus, if we denote the period of x by p, there exists k with $0 \leqslant k < p$ and $f^k(x) \in U_2$. This implies $f^m p + k(x) \in U_2$ for any positive integer m. Now choose m such that $l = mp + k \geqslant n_0$. Then we have $g_1(V_1) \cap V_2 \neq \varnothing$ and there exists a point $y \in V_1$ with $g_1(y) \in V_2$. Now, for the point $(x, y) \in U_1 \times V_1$ we get $(f \times g)_l(x, y) \in U_2 \times V_2$. Hence $f \times g$ is topologically transitive. □

Example 5.3.3. Some of the best-known chaotic maps are:

1. The logistic map: f : [0, 1] → [0, 1], f(x) = 4x(1 - x)
2. The map doubling the circle: D : S1 → S1 D(θ)=2θ
3. The baker map: B : [0, 1] → [0, 1], B(x) = $\begin{cases} 2x & , if\, 0 \leqslant x < 1/2 \\ 2x-1 & , if\, 1/2 \leqslant x \leqslant 1. \end{cases}$

Theorem 5.3.7. [45] If $f^\star_\infty$ or (resp. $f^\star_N$) is Devaney chaotic, then each factor map f_i is also Devaney chaotic. However, the converse does not hold

[46]. For any fixed positive integer i, it follows from definition of Devaney Chaos, that under $f^\star_\infty$, for any pair of not empty subsets A_i, B_i of X_i, there exists some

integer $n > 0$, such that:

$$(f_\infty^\star)^n \left(A_i \times \prod_{j\neq i} X_j\right) \cap \left(B_i \times \prod_{j\neq i} X_j\right) = (f_i^n(A_i) \cap (B_i)) \times \prod_{j\neq i} f_j^n(X_j) \neq \varnothing$$

So, $f_i^n(A_i) \cap B_i \neq \varnothing$, this implies that f_i is transitive. Clearly, $\overline{Per(f_i)} = X_i$ as $\prod_{i=1}^{\infty} X_i = \overline{Per(f_\infty^\star)} \subset \prod_{i=1}^{\infty} \overline{Per(f_i)}$ ($f_\infty^\star$ is Devaney chaotic). Thus, for each i , f_i is topologically transitive and has denseness of periodic points, by *Theorem 4.1*, f_i is Devaney chaotic. A similar argument can be given when we take a finite family of dynamical systems. □

Theorem 5.3.8 ([46])**.** Let $X_i = [0, 1]$ for each $i \in \mathbb{N}$. Then $f_\infty^\star$ (resp. $f_N^\star$) is Devaney Chaotic if and only if $f_\infty^\star$ (resp.$f_N^\star$) is transitive.

Proof. ($\Leftarrow$) Since $f_\infty^\star$ is transitive, and by the above theorem, we know that for each i, f_i is transitive as well as chaotic in the sense of Devaney. Thus,
$\prod_{i=1}^{\infty} X_i = \prod_{i=1}^{\infty} \overline{Per(f_i)} = \overline{\prod_{i=1}^{\infty} Per(f_i)}$ Also, we know that
$\sup\{\min P(f_i) : i \in \mathbb{N}\} = 1 < infty$ holds since each f_i has a fixed point. Also by *Definition 1.4.3*, $\overline{Per(f_\infty^\star)} = \overline{\prod_{i=1}^{\infty} Per(f_i)} = \prod_{i=1}^{\infty} X_i$.
Thus, $f_\infty^\star$ has dense periodic points and is chaotic by *Theorem 4.1.*
($\Rightarrow$) Now, this proof holds trivially since any chaotic map is also transitive. Our proof is complete. □

Theorem 5.3.9 ([48])**.** Let $f : X \to X$ be a continuous chaotic cascade on the metric space X and $g : Y \to Y$ be another cascade on the metric space Y. If g is topologically mixing and the set of all periodic points of $f \times g$ is dense in $X \times Y$, then $f \times g : X \times Y \to X \times Y$ is chaotic.

Proof. Now, in order to prove this, we will prove transitivity as well as sensitivity of $f \times g$.

Transitivity: For any nonempty sets $W_1, W_2 \subset X \times Y$, there exists nonempty subsets

$U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ with $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$ and clearly

$$N_{f\times g}(U_1 \times V_1, U_2 \times V_2) = N_f(U_1, U_2) \bigcap N_g(V_1, V_2)$$

where $N_f(U, V)$ is as defined in *Definition 1.3.9.* As g is topologically mixing , there is $M > 0$ such that

$$[M, \infty) \subset N_g(V_1, V_2)$$

Since f is continuous, $f^-M(U_2)$ is a nonempty and open subset of X.

And if f is topologically transitive, by definition, there exists an $n \in \mathbb{N}$ such that

$$N_f(U_1, f^{-Mn}(U_2)) \neq \varnothing$$

which implies that

$$N_f(U_1, U_2) \bigcap N_g(V_1, V_2) \neq \varnothing$$

Thus, the product semi-flow $f \times g$ is topologically transitive.

Sensitivity: Assume, on the contrary, that both f and g are not sensitive. This means that for any $\epsilon > 0$, there exists an $x \in X$ such that for a particular open set $U \subset X$ with $x \in U$, $d_X(f^n(x), f^n(x')) \leqslant \frac{\epsilon}{2}$, for any $x' \in U$ and any $n \in \mathbb{N}$. Similarly, there is a $y \in Y$ such that for a certain open set $V \subset Y$ with $y \in V$, $d_Y(g^n(y), g^n(y)) \leqslant \frac{\epsilon}{2}$ for any $y' \in V$ and any $n \in \mathbb{N}$. So, $N_{f\times g}(U \times V, \epsilon) = \varnothing$. Thus, $g \times g$ is not sensitive, which is a contradiction to our hypothesis of it being sensitive. Hence proved that $f \times g$ is chaotic. □

Theorem 5.3.10 ([48]). Let X be a metric space and assume that $f : X \to X$ is a continuous cascade with Touhey property. Let $g : Y \to Y$ be a non-necessarily continuous, chaotic, and topologically mixing cascade on the metric space Y. If the set of all periodic points of $f \times g$ is dense in $X \times Y$, then $f \times g : X \times Y \to X \times Y$ is chaotic.

Proof. By definition, it is enough to show that $f \times g : X \times Y \to X \times Y$ is topologically transitive. Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset V$ be nonempty open sets. Since g is topologically mixing, there $n_0 \in \mathbb{N}$ with $g^n(V_1) \bigcap V_2 \neq \varnothing$ and all $n \geqslant n_0$. By definition, there exists a periodic point $x \in U_1$ whose orbit U_2. let M be the period of x. Then there exists M' with $0 \leqslant M' < M$ and $f^{M'}(x) \in U_2$. This means that $f^{kM+M'}(x) \in U_2$ for any integer $k > 0$. Choose $k > 0$ such that $M'' = kM + M' \geqslant n_0$. Therefore, there exists a point $y \in V_1$ with $g^{L''}(y) \in V_2$. Consequently, $(x, y) \in (f \times g)^{M''}(U_1 \times V_1) \cap (U_2 \times V_2)$, which implies that $f \times g : X \times Y \to X \times Y$ is topologically transitive. Hence, the proof is complete. □

Theorem 5.3.11 ([53]). Let (X, f) and (Y, g) be 2 spaces with their corresponding maps, If $(X \times Y, f \times g)$ is Poincaré chaotic then at least one of (X, f);(Y, g) is Poincaré chaotic.

Proof. Let (x, y) be an unpredictable transitive point in the semiflow $(X \times Y, f \times g)$ with unpredictability constant c. Clearly, x and y are transitive points of semiflows (X, f) and (Y, g) respectively. By definition of unpredictable points, there exist sequences t_n and s_n, both of which diverge to infinity, such that $t_n(x, y) \to (x, y)$ and $d[s_n(x, y).(s_n + t_n)(x, y)] \geqslant c$ for each $n \in \mathbb{N}$ which implies that $t_n x \to x$ and $t_n y \to y$. We claim that at least one of x, y s unpredictable points with unpredictability constant $c/2$. Now, consider the following cases:

Case 1. If $d_1[s_n x, (s_n + t_n)x] \geqslant c/2$ for each $n \in \mathbb{N}$ or $d_2[s_n x, (s_n + t_n)x] \geqslant c/2$ for each $n \in \mathbb{N}$, our proof is complete.

Case 2. If $\{n \in \mathbb{N} : d_1[s_n x, (s_n + t_n)x] < c/2\}$ is nonempty but finite, say $n_1, n_2, \ldots, n_k$. Choose a fixed natural number n_0 such that $n_0 > n_i$ for each $i \in \{1, 2, \ldots, k\}$. Now, consider 2 sequences u_n, v_n where $u_n = s_n, v_n = t_n$ for all $n \notin n_1, n_2, \ldots, n_k$ and $u_n = s_n, v_n = t_n$ for each $n \in \{n_1, n_2, \ldots, n_j\}$. Clearly, we can verify that u_n and v_n both diverge to infinity such that $v_n x \to x$ and $d_1[u_n x, (s_n + t_n)x] \geqslant c/2$ for each $n \in \mathbb{N}$. Thus, x is an unpredictable point and hence our claim is proved.

Case 3. If $A = \{n \in \mathbb{N} : d_1[s_n x, (s_n + t_n)x] \geqslant c/2\}$ is infinite then we can write A as $n)1, n_2, hdots$ where $n_k < n_{k+1}$ for all $k \geqslant 1$. Note that the sequences $s_{n_k k \in \mathbb{N}}, t_{n_k k \in \mathbb{N}}$, both diverge to infinity and $t_{n_k} y \to y$. Also, $d_2[s_{n_k} y, (s_{n_k} + t_{n_k})y] > c/2$ for each $k \in \mathbb{N}$ because if $d_2[s_{n_k} y, (s_{n_k} + t_{n_k})y] \leqslant c/2$ for some $k' \in \mathbb{N}$, then

$$d[s_{n'_k}(x, y), (s_{n'_k} + t_{n'_k})(x, y)] = d_1[s_{n'_k} x, (s_{n'_k} + t_{n'_k})x] + d_2[s_{n'_k} y, (s_{n'_k} + t_{n'_k})y] < c/2 + c/2 = c$$

which is a contradiction to the unpredictability of (x, y). Hence, y is an unpredictable point with unpredictability constant $c/2$ and hence our claim is proved. □

Theorem 5.3.12 ([53]). If (X_∞, f_∞) is strongly Auslander-Yorke chaotic then there exists a positive integer k such that (X_k, f_k) is strongly Auslander-Yorke chaotic.

Proof. Now, since (X_∞, f_∞) is topologically transitive, we know by factor maps that the factors are also topologically transitive, so, for all i, (X_i, f_i) is topologically transitive. Similarly, using the properties, sensitivity of (X_∞, f_∞) implies sensitivity of at least one of the factors, say (X_k, f_k).

Now, we prove that (X_k, f_k) has a dense set of recurrent points. Let U_k be a nonempty open subset of X_k. Since (X_∞, f_∞) has a dense set of recurrent points, there exists a recurrent point $(x_i) \in \prod_{i=1}^{\infty} V_i$ where $V_i = X_i$ for all $i \neq k$ and $V_k = U_k$. By definition of a recurrent point, there exists an increasing sequence $\{t_n\}$ diverging to infinity such that $t_n(x_i) \to (x_i)$ which implies that $t_n x_k \to x_k$ and hence $x_k \in U_k$ is a recurrent point. Thus, (X_k, f_k) has a dense set of recurrent points. Thus, (X_k, f_k) is strongly Auslander-Yorke chaotic. □

Theorem 5.3.13 ([53]). If (X_∞, f_∞) is strongly Ruelle-Takens chaotic then there exists a positive integer k such that (X_k, f_k) is strongly Ruelle-Takens chaotic.

Proof. Let (x_i) be a transitive point of the semiflow (X_∞, f_∞). For any positive integer j, let U_j be a nonempty open subset of X_j. Since (x_i) is a transitive point of the semiflow (X_∞, f_∞), there exists an $n \in \mathbb{N}$ such that $t_n(x_i) \in \prod_{i=1}^{\infty} V_i$ where $V_i = X_i$ for all $i \neq j$ and $V_j = U_j$. This implies that $t_n x_j \in U_j$. Thus, x_j is a transitive point of the semiflow (X_j, f_j). Now, from the above proof, we can say that there exists a positive integer k such that (X_k, f_k) has a dense set of recurring points and hence (X_k, f_k) is strongly Ruelle-Takens chaotic. □

We know from Example 2 that the converse is not true.

Now, it is naturally asked: What is the necessary and sufficient condition in Theorem 3.4.4. We shall give a partial answer to this question by using the following two lemmas:

Lemma 16 ([53]). *f_N^* is mixing if and only if f_i is mixing for each $1 \leqslant i \leqslant N$.*

Proof. $(\Leftarrow)$ It holds trivially.

$(\Rightarrow)$ Given any fixed $1 \leqslant i \leqslant N$, for any pair of non-empty open subsets A_i, $B_i \subset X_i$, since f_N^* is mixing, we have that there exists some positive integer m such that for any $n > m$,

$$\left((f_N^*)^n(A_i \times \prod_{j\neq i} X_j)\right) \cap \left((B_i \times \prod_{j\neq i} X_j)\right) = (f_i^n(A_i) \cap B_i) \times \prod_{j\neq i} f_j^n(X_j) \neq \varnothing.$$

This implies that $f_i^n(A_i) \cap B_i \neq \varnothing$, i.e., f_i is mixing. □

Lemma 17 ([45]). *f_∞^* is mixing if and only if the factor map f_i is mixing for each $i \in \mathbb{N}$.*

Proof. $(\Leftarrow)$ For any pair of non-empty open sets $A, B \subset X_{(\infty)}$, according to the construction of open sets of $X_{(\infty)}$, it follows that there exists a positive integer n^* and non-empty open subsets $A_i, B_i \subset X_i$ such that

$$\prod_{i=1}^{n^*} A_i \times \prod_{i>n^*} X_i \subset A \quad \text{and} \quad \prod_{i=1}^{n^*} B_i \times \prod_{i>n^*} X_i \subset B.$$

For each integer $i > 0$, since f_i is mixing, then there exists some positive integer N_i

such that $f_i^n(A_i) \cap B_i \neq \varnothing$ for each $n > N_i$. Put $N^* = \max\{N_i : 1 \leqslant i \leqslant n^*\}$. It is not difficult to check that

$$(f_\infty^*)^n(A) \cap B \supset \left(\prod_{i=1}^{n^*}(f_i^n(A_i) \cap B_i)\right) \times \prod_{i>n^*} f_i^n(X_i) \neq \varnothing$$

holds for each $n \geqslant N^*$. So f_∞^* is mixing.

($\Rightarrow$) Similarly to the proof of Lemma 4, it holds trivially. □

Theorem 5.3.14 ([45]). Assume that (X, d) is a compact metric space. If the system (X, f) is chaotic in the sense of Devaney, then $Per(f)$ is infinite.

Proof. Clearly, the set X is infinite. Suppose that $Per(f)$ is finite. Without loss of generality, we may assume $P(f) = \{n_1, n_2, \ldots, n_k\}$. It is easy to see that $f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(p) = p$ holds for any $p \in \mathrm{Per}(f)$.

Now we assert that for any $x \in X$, $f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(x) = x$, i.e., $\mathrm{Per}(f) = X$. Indeed, choosing arbitrarily $x \in X$, it follows from $\mathrm{Per}(f) = X$ that there exists a sequence $\{p_i\}_{i=1}^\infty \subset \mathrm{Per}(f)$ such that $\lim_{n\to\infty} d(p_n, x) = 0$. Since $f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}$ is uniformly continuous, then we have

$$\lim_{n\to\infty} d(p_n, f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(x)) = \lim_{n\to\infty} d(f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(p_n), f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(x)) = 0.$$

This implies that $f^{n_1 \cdot n_2 \cdot \ldots \cdot n_k}(x) = \lim_{n\to\infty} p_n = x$.

We know from the transitivity of f that there exists $x_0 \in X = \mathrm{Per}(f)$ such that $\mathrm{orb}_f(x_0) = X$, so X is finite, which is a contradiction. □

Theorem 5.3.15 ([55]). The product dynamical system $(X \times Y, f \times g)$ is Li-Yorke chaotic if one of the dynamical systems (X, f) or (Y, g) is Li-Yorke chaotic.

Proof. Suppose (X, f) is Li-Yorke chaotic. By the definition of Li-Yorke chaos, (X, f) has an uncountable scramble set S. Let $y \in Y$ be any point. Consider the subset $S \times \{y\}$ of $X \times Y$. It is obvious that $S \times \{y\}$ is uncountable. Let $u = (x, y)$

and $v = (z, y)$ be two points of $S \times \{y\}$. $d(u, v) = d_1(x, z) + d_2(y, y) = d_1(x, z)$. Since (X, f) is Li-Yorke chaotic, we have $\lim_{n\to\infty} d_1(f^n(x), f^n(z)) = 0$, and consequently,

$$\lim_{n\to\infty} d(f \times g^n(x, y), f \times g^n(z, y)) = \lim_{n\to\infty} [d_1(f^n(x), f^n(z)) + d_2(g^n(y), g^n(y))] = 0.$$

Also, $\limsup_{n\to\infty} d_1(f^n(x), f^n(z)) > 0$, therefore,

$$\limsup_{n\to\infty} d(f \times g^n(x, y), f \times g^n(z, y)) = \limsup_{n\to\infty} [d_1(f^n(x), f^n(z)) + d_2(g^n(y), g^n(y))] > 0.$$

Thus, $S \times \{y\}$ is an uncountable scramble set, therefore, $(X \times Y, f \times g)$ is Li-Yorke chaotic. $\square$

Theorem 5.3.16 ([55]). The product dynamical system $(X \times Y, f \times g)$ has positive topological entropy, then one of the dynamical systems (X, f) or (Y, g) is Li-Yorke chaotic.

Proof. We know that the topological entropy is non-negative. Given that $h(f \times g) > 0$, and since $h(f \times g) = h(f) + h(g)$, we have either $h(f) > 0$ or $h(g) > 0$, or both $h(f)$ and $h(g)$ are greater than 0. Hence, at least one of the dynamical systems is Li-Yorke chaotic.

We also know that if $A \times B$ is uncountable, then at least one of the sets A or B is uncountable. Suppose $A \times B$ is an uncountable scramble subset of $X \times Y$. For $(x_1, y_1), (x_2, y_2) \in A \times B$, we have

$$\begin{aligned}
&\lim_{n\to\infty} d(f \times g^n((x_1, y_1), (x_2, y_2))) = 0 \\
&\Rightarrow 0 \leqslant \lim_{n\to\infty} d_1(f^n(x_1), f^n(x_2)) + \lim_{n\to\infty} d_2(g^n(y_1), g^n(y_2)) \\
&\Rightarrow \lim_{n\to\infty} d_1(f^n(x_1), f^n(x_2)) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_2(g^n(y_1), g^n(y_2)) = 0.
\end{aligned}$$

Also,

$$\limsup_{n\to\infty} d(f \times g^n(x_1, y_1), f \times g^n(x_2, y_2))$$

$\Rightarrow$ either $\limsup_{n\to\infty} d_1(f^n(x_1), f^n(x_2)) > 0$ or $\limsup_{n\to\infty} d_2(g^n(y_1), g^n(y_2)) > 0$ or both > 0. $\square$

This proves that at least one of the sets A or B is an uncountable scramble set.

Conclusion

In this chapter we studied various properties and results concerning product of dynamical systems. In **Theorem 5.1.2** and **Theorem 5.1.3** we analysed how properties from dynamical systems are carried over to their product. In **Theorem 5.3.1** we proved that some forms of sensitivity can be inherited by the product dynamical system from its constituent dynamical systems.We also listed classical examples of chaotic maps in **Example 5.3.3**. Additionally, we proved various sufficient conditions like **Theorem 5.3.8** and **Theorem 5.3.9** under which, the product of two chaotic cascades is chaotic.

Bibliography

[1] Stephen Silverman On Maps with Dense Orbits and the Definition of Chaos. Rocky Mountain Journal of Mathematics(1992), 22(1), April 1992.

[2] Jie-Hua Mai and Wei-Hua Sun Almost periodic points and minimal sets in ω-regular spaces Topology and its Applications(2007) Vol 154(15) Pg 2873–2879

[3] N. Değirmenci and Ş. Koçak Existence of a dense orbit and topological transitivity: When are they equivalent Acta Mathematica Hungarica(2003) Vol 99(3) Pg 185–187

[4] Heriberto Roma n-Flores. A note on transitivity in set-valued discrete systems. Chaos, Solitons and Fractals, 17(1):99–104, 2003.

[5] Ethan Akin and Joseph Auslander and Anima Nagar. Studia Mathematica Vol 235(3) 225–249

[6] Garnett P. Williams , Chaos Theory Tamed (1989) , Pg19

[7] Robert S. P. Beekes, Etymological Dictionary of Greek (Leiden: Brill, 2009), 2:1614, Pg 1616–17.

[8] Hesiod, Theogony, Line no 116.

[9] https://timesofindia.indiatimes.com/Order-chaos-and-creation/articleshow/10552328.cms

[10] Christian Oestreicher , A History of Chaos Theory, Dialogues in Clinical Neuroscience(2007), Vol 9:3, Pg 279-289,

[11] LorenzEN., Deterministic non periodic flow, AMSJ.(1963), vol 20 , Pg 130-141.

[12] Mandelbrot B, Formes nouvelles du hasard dans les sciences, Économie Appliquée (1973), Vol 26, Pg 307-319

[13] John Guckenheimer, Sensitive dependence on initial conditions for one-dimensional maps, Commm. Math Phys (1979), Vol 70 , Pg 133-60

[14] R. Devaney, An Introduction to Chaotic Dynamical Systems(1989), Addison–Wesley, Pg 60

[15] Akin E, Kolyada S., Li–Yorke sensitivity, Nonlinearity (2003), Vol - 16(4), Pg 1421–33

[16] Xiong J., Chaos in a topologically transitive system. , Sci China Ser A (2005), Vol 48(7), Pg 929–39.

[17] T.K. Subrahmonian Moothathu, Stronger forms of sensitivity for dynamical systems, Nonlinearity(2007), Vol 20(9), Pg 2115-2126

[18] George Cantor, Begründung der transfiniten Mengenlehre, Mathematische Annalen(1897), Vol 49, Pg 207-246,

[19] Tej Bahadur, Elements of Topology(2013) , CRC Press, pg 1

[20] Hunt, B and Kan, I and Yorke, J Intersection of thick Cantor sets, Trans. Amer. Math. Soc(1993) Vol-339 Pg 869 888

[21] Liu, Heng and Liao, Li and Wang, Lidong and others, Thickly syndetical sensitivity of the topological dynamical system, Discrete Dynamics in Nature and Society(2014), Vol 2014

[22] Wu, Xinxing and Ma, Xin and Chen, Guanrong and Lu, Tianxiu, A note on the sensitivity of semiflows, Topology and its Applications(2020), Vol 271 , Pg 107046

[23] Li, J., Oprocha, P., Wu, X. Furstenberg families, sensitivity and the space of probability measures, Nonlinearity(2017), Vol 30(3), Pg 987–1005

[24] Sharma, Puneet and Nagar, Anima, Inducing sensitivity on hyperspaces, Topology and its Applications(2010), Vol 157(13), Pg 2052 - 2058,

[25] Banks J, Brooks J, Cairns G, Davis G and Stacey P 1992 On Devaney's, definition of chaos Am. Math. Mon., Vol 99, Pg 332–4

[26] Blanchard F et al (ed) , Topics in Symbolic Dynamics and Applications(2000), Vol 279

[27] Blanchard F, Glasner E, Kolyada S and Maass A , On Li–Yorke pairs Walter de Gruyter GmbH and Co. KG Berlin, Germany (2002),

[28] D. Ruelle and F. Takens, "Note concerning our paper: "On the nature of turbulence" Communications in Mathematical Physics(1971), Vol 23, Pg 343–344

[29] D. Ruelle and F. Takens, "On the nature of turbulence", Communications in Mathematical Physics, Vol 20 , Pg 167–192

[30] G. F. Liao, L. D. Wang, and Y. C. Zhang, "Transitivity, mixing and chaos for a class of set-valued mappings," Science in China A: Mathematics(2006), Vol. 49(1), Pg. 1–8

[31] Ruette, Sylvie, Chaos on the interval-a survey of the relationship between the various kinds of chaos for continuous interval maps, arXiv preprint arXiv:1504.03001(2015)

[32] Akin E, Auslander J and Berg K 1996 , When is a transitive map chaotic? , Convergence in Ergodic Theory and Probability(1996), Vol 5, Pg 25–40

[33] Miller, Alica, A note about various types of sensitivity in general semiflows, Applied general topology(2018), Vol 19(2), Pg 281 - 289

[34] Kurka P 2003 , Topological and Symbolic Dynamics , Sociét é Math ématique dé (2003), Vol 11

[35] Petersen K E A topologically strongly mixing symbolic minimal set Trans. Am. Math. Soc.(1970) Vol 148 Pg 603–12

[36] Birkhoff, George David. Dynamical systems. Vol. 9. American Mathematical Soc., 1927., pp. 1

[37] Brin, Michael, and Garrett Stuck. Introduction to dynamical systems. Cambridge university press, 2002. pp. 1-2

[38] Money, Chad. "Chaos in semiflows." (2015), pp. 2-16

[39] Banks, John, et al. "On Devaney's definition of chaos." The American Mathematical Monthly 99.4 (1992): 332-334.

[40] Devaney, Robert. An introduction to chaotic dynamical systems. CRC Press, 2018. pp. 50

[41] S. Wiggins, "Introduction to Applied Nonlinear Dynamical Systems and Chaos, Second Edition", Springer, 2003, pp. 736-737

[42] Guirao, Juan Luis García, and Marek Lampart. "Relations between distributional, Li–Yorke and ω chaos." Chaos, Solitons and Fractals 28.3 (2006): 788-792.

[43] Robinson, Clark. Dynamical systems: stability, symbolic dynamics, and chaos. CRC Press, 1998., pp. 84-89

[44] Effah-Poku, S., William Obeng-Denteh, and I. K. Dontwi. "A study of chaos in dynamical systems." Journal of Mathematics 2018 (2018).

[45] Vellekoop, Michel, and Raoul Berglund. ”On intervals, transitivity = chaos.” The American Mathematical Monthly 101.4 (1994): 353-355.

[46] Wu, Xinxing, and Peiyong Zhu. ”Devaney chaos and Li-Yorke sensitivity for product systems.” Studia Scientiarum Mathematicarum Hungarica 49.4 (2012): 539-545

[47] Kreyszig, Erwin. Introductory functional analysis with applications. Vol. 17. John Wiley and Sons, 1991., pp. 18, 77

[48] Li, Risong, and Xiaoliang Zhou. ”A note on chaos in Product of Dynamical Systems.” Turkish Journal of Mathematics 37.4 (2013): 665-667, 672-673

[49] Huang, Wen, and Xiangdong Ye. ”Devaney’s chaos or 2-scattering implies Li–Yorke’s chaos.” Topology and its Applications 117.3 (2002): 260, 262, 268

[50] Nagashima, Hiroyuki. Introduction to chaos: physics and mathematics of chaotic phenomena. CRC Press, 2019, pp. 1-4

[51] Barwell, Andrew David. Omega-limit sets of discrete dynamical systems. Diss. University of Birmingham, 2011., pp. 12-13

[52] Miller, Alica. ”Unpredictable points and stronger versions of Ruelle–Takens and Auslander–Yorke chaos.” Topology and its Applications 253 (2019): 7-16.

[53] Thakur Rahul, and Ruchi Das. ”Strongly Ruelle-Takens, strongly Auslander-Yorke and Poincaré chaos on semiflows.” Communications in Nonlinear Science and Numerical Simulation 81 (2020): 105018.

[54] Syahida Che Dzul-Kifli and Chris Good The American Mathematical Monthly Vol 122(8) Pg 773

[55] Khundrakpam Binod Mangang, ”Li- Yorke Chaos in Product Dynamical Systems” , Advances in Dynamical Systems and Applications.ISSN 0973-5321, Volume 12, Number 1, (2017) pp. 81-88 © Research India Publications

[56] Degirmenci, Nedim and Kocak, Sahin (2010), ”Chaos in Product of Dynamical Systems”, Turkish Journal of Mathematics: Vol. 34: No. 4, Article 14.

[57] Kolyada, Sergiy and Snoha, L’ubomír, Grazer Mathematische Berichte (1997), Vol 334, Pg 3-35