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"Study of the soliton solutions of nonlinear partial differential equations using Unified method"

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SRIVIPRA PROJECT 2023

Title: Study of the soliton solutions of nonlinear partial differential equations using Unified method

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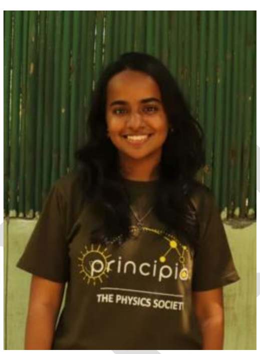
6		Gayathri Pillai	1821015	B. Sc. (H) Physics	Gayathri

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Certificate of Originality

This is to certify that the aforementioned students from Sri Venkateswara College have participated in the summer project SVP-2023 titled "Study of the soliton solutions of nonlinear partial differential equations using Unified method". The participants have carried out the research project work under my guidance and supervision from 15 June, 2023 to 15th September 2023. The work carried out is original and carried out in a hybrid mode.

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Abstract

In this project, we study the applications of unified method to the nonlinear evolution equations which represent some of the important physical phenomenon. Partial differential equations (PDEs) can be regarded as evolution equations on an infinite dimensional state space. So, our primary objective in this project is to identify the symmetries of some nonlinear partial differential equations in order to obtain solitary wave solutions. Furthermore, to discuss the dynamic behavior of obtained solution for the equations under consideration. The investigations carried out in this dissertation are confined to the applications of method for three nonlinear partial differential equations such as the (1+1)-dimension Lonngren Equation, the (1+1)-dimensional Burger's Equation, the (1+1)-dimensional longitudinal wave equation,
Our project comprises four chapters. In **Chapter 1**, some important features of a unified method reviewed which are of great importance to the work dealt in Chapters 2–4. It also presents the methodologies utilized in the dissertation and a brief account of the related studies made by various authors in this field of research. The chapter-wise description is as follows:

Chapter 2, 3, and 4 deals with the study of we employ the unified method to derive solitary wave solutions for the the (1+1)-dimension Lonngren Equation, the (1+1)-dimensional Burger's Equation, the (1+1)-dimensional longitudinal wave equation, and the (1+1)-dimensions longitudinal wave equation, presenting a systematic approach to reveal the underlying dynamics. By incorporating mathematical analysis and numerical simulations, we investigate the behaviors and properties of these solitary wave solutions in various fields of science. These obtained solutions shed light on the fundamental mechanisms governing solitary waves of the longitudinal wave equation, contributing to a deeper understanding of nonlinear wave phenomena. We present effective visualizations of the dynamical wave structures necessary in the obtained solutions to improve our comprehensive understanding. We visualize their significance using various graphs,

such as 3D, 2D, and contour plots to demonstrate these obtained solutions. Consequently, numerous types of wave profiles, including singular periodic, multi-periodic, bell-shaped, traveling wave, and multi-bell-shaped wave profiles were found. In summary, this work presents a comprehensive study of solitary wave solutions using the unified method for the (1+1)-dimensions equation. The outcomes of the current study manifest that the considered method is significant and systematic in solving nonlinear evolution equations.

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Chapter 1

Literature review and Introduction

1.1. Background and inspiration

The study of nonlinear differential equations has played an essential role in real-world problems because it has been used as a model to describe complex physical phenomena in almost every field of science and engineering for nearly the last decades, especially in plasma physics, fluid mechanics, solid-state physics, and plasma wave. Most of the problems are nonlinear in the real world and are often represented by a system of differential equations or a single differential equation. It is challenging to conceive any area of applications where its effect is not felt. It is important to obtain their numerical and exact analytical solutions to understand these phenomena better and use them in practical scientific research. Generally, the nonlinear partial differential equations are still complicated to solve numerically and theoretically. However, a significant arrangement of action has been disbursed over the last ten decades or so in attempting to find powerful, robust, and stable numerical and analytical methods for solving nonlinear partial differential equations of physical interest.

A differential equation is a mathematical equation that involves one or more derivatives of an unknown function. These equations are used to describe how a function and its derivatives change with respect to one or more independent variables. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions. Many techniques and methods, such as separation of variables, integrating factors, and numerical methods like Euler's method or finite element methods, are employed to solve differential equations. Many real-world phenomena can be modeled mathematically by using differential equations.

(1) **Ordinary Differential Equations (ODE's):** ODE's involve a single independent variable and its derivatives. They describe processes or phenomena evolving

with respect to a single variable, such as time. Common examples include Newton's second law of motion and radioactive decay. They can be solved by various processes, like the separation of variables with the help of integrating factors, etc.

(2) **Partial Differential Equations (PDE's):** PDE's involve multiple independent variables and their partial derivatives. PDEs are used to describe phenomena that depend on multiple variables, such as heat conduction, fluid flow, and wave propagation. For example, the heat equation, wave equation, and Schrödinger equation in quantum mechanics They can be solved by the separation of variables, canonical forms, etc.

(3) **Linear Differential Equations:** These are differential equations where the unknown function and its derivatives appear in a linear manner. Linear differential equations are typically more amenable to analytical solutions, and many real-world problems can be approximated as linear under certain conditions.

(4) **Non-Linear Partial Differential Equations:** In these, the unknown function and its derivatives appear in a nonlinear manner. Nonlinear differential equations are generally more challenging to solve analytically, and their solutions often require numerical techniques or approximations.A general ordinary differential equation can be written concisely in the form:

$$F(t, u, \frac{du}{dt}) = 0,$$

involving the continuous Independent and dependent variable. As,

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y).$$

is a linear (PDE) where coefficients of u, u_x, u_y are functions of independent variable x, y. A (PDE) is said to be nonlinear if the relations between the unknown functions and their partial derivatives involved in the equation are nonlinear. The distinction between a linear and a nonlinear partial differential equation is usually made in terms of the properties of the operator that defines the (PDE's) itself.

1.2. Applications of Differential Equation:

(1) **Application of first order differential equation: Orthogonal trajectory** An orthogonal trajectory is a curve that cuts each member of a family of curves at a right angle. Or, to put it another way, it is the family of curves that cross another family of curves perpendicularly. We use the differential equations in

order to solve cartesian and polar curves and find their respective orthogonal trajectories.

(2) **Application to tuned mass dampers (TMD)**:A single-story shear building vibrates in the event of an earthquake. The shear building is a rigid girder of mass m and columns of combined stiffness k. The horizontal displacement $x(t)$ can be used to model the girder's vibration. The ground displacement $x_0(t)$, as depicted, is used to mimic the earthquake. The internal friction between different building parts causes the girder to vibrate with a damping force, which is represented by the equation $cx'(t) - x_0'(t)$, where c is the damping coefficient. The equation controls the relative displacement $y(t)$, $x(t)$, $x_0(t)$ between the girder and the ground according to a second order linear ordinary differential equation:

$$m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + ky(t) = -m\frac{\partial^2 x_0}{\partial t^2}.$$

.

1.3. Dimension of a Partial Differential Equations(PDE's)

(1+1)-dimensional PDE's: In a (1+1)-dimensional setting, nonlinear differential equations involve one spatial variable (x) and one temporal variable (t) and contain nonlinear terms. Here is an example of a $(1+1)$ dimensional nonlinear differential equation:

$$\frac{\partial u}{\partial t} = F(x, t, u, \frac{\partial u}{\partial x}).$$

where F is a nonlinear function that relates the variables $x, t, u, \frac{\partial u}{\partial x}$. There are numerous types of nonlinear differential equations that can be encountered, and their behavior and solution techniques depend on their specific forms. Some notable examples include: Wave Equation, Diffusion Equation, Schrø"dinger Equation.

(2+1)-dimensional PDE's: A $(2+1)$-dimensional nonlinear partial differential equation (PDE) is a mathematical equation that involves functions of two spatial dimensions x and y and one time dimension (t), and it includes nonlinear terms, meaning that the dependent variable and its derivatives are involved in nonlinear combinations. These equations are commonly used to model complex physical phenomena. The general form of a $(2+1)-$dimensional nonlinear PDE is:

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}\right).$$

Some notable examples include: The Korteweg-de Vries Equation (KdV), The Fisher-KPP Equation (Fisher's Equation), The Nonlinear Schrø"dinger Equation.

(3+1)-dimensional PDE's: A $(3+1)$-dimensional nonlinear partial differential equation (PDE) is an equation that involves derivatives with respect to three spatial dimensions (x, y, z) and one time dimension (t), and it contains nonlinear terms. These equations describe how physical quantities change over both space and time and often arise in various fields of science and engineering.

This project intends to confer $(1+1)$-Dimensional Partial Differential Equation. These are used to mathematically formulate, and thus aid the solution of, physical and other problems involving functions of several variables, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics, etc.

1.4. Some important Partial Differential Equations(PDE's)

Korteweg-de Vries (KdV) The Korteweg-de Vries (KdV) equation is a partial differential equation (PDE's) that describes the evolution of long, one-dimensional waves in certain dispersive systems. The KdV equation was first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries (1895), who found the simplest solution, the one-soliton solution. Understanding of the equation and behavior of solutions was greatly advanced by the computer simulations of Zabusky and Kruskal in 1965 and then the development of the inverse scattering transform in 1967. The KdV equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0.$$

where $u(x, t)$ represents the amplitude of the wave at position x and time t. The coefficients c and β depend on the specific system being modeled. The first term on the left-hand side of the equation $(\frac{\partial u}{\partial t})$ describes the time evolution of the wave. The second term $(c\frac{\partial u}{\partial x})$ accounts for the advection of the wave with a velocity c. The third term $(\beta\frac{\partial^3 u}{\partial x^3})$ represents the dispersive effects of the system, allowing the wave to disperse or spread out over time.

Kadomtsev-Petviashvili (KP) The Kadomtsev-Petviashvili (KP) equation is a partial differential equation that describes certain types of nonlinear waves in two spatial dimensions. It was introduced independently by B.B. Kadomtsev and V.I. Petviashvili in the late 1960s. The KP equation is particularly relevant in the study of water waves and plasma physics. The KP equation is given by:

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0,$$

where u is a scalar function of two independent variables x and y, t is the time variable, ut denotes the partial derivative of u with respect to t, u_x and u_{xx} denote the partial derivatives of u with respect to x and their second order, respectively, and similarly, u_{yy} denotes the partial derivative of u with respect to y and u_{xxx} denotes the partial derivative of u with respect to x and its third order. The coefficient σ determines the strength of the dispersion in the y direction and $\sigma^2 = \pm 1$. The case

$$(u_t + u_{xxx} + 6uu_x)_x - 3u_{yy} = 0,$$

is known as the KPII equation, and models, for instance, water waves with small surface tension. The case $\sigma = i$ is known as the KPI equation, and may be used to model waves in thin films with high surface tension.employed to study its behalf the same way that the KdV equation can be viewed as a universal integrable system in one spatial dimension, the KP equation is a universal integrable system in two spatial dimensions because many other integrable systems can be obtained as reductions. As a result, for the past forty years, the KP equation has been the subject of intense research in the mathematical community. The KP equation, which results from the reduction of a system with quadratic nonlinearity that admits weakly dispersive waves in a paraxial wave approximation, is also one of the most widely used models in nonlinear wave theory. In the asymptotic description of such systems, where only the leading order terms are kept and an asymptotic equilibrium between weak and strong terms is achieved, the equation arises naturally as a distinct limit. The KP equation is a fundamental equation in soliton theory, as it supports soliton solutions. Solving the KP equation analytically can be challenging due to its nonlinear and dispersive nature. However, numerical methods and approximations are often vior and obtain solutions for specific cases.

Non-linearity of KP equation: The KP equation is a nonlinear dispersive equation, meaning that it exhibits both nonlinear and dispersive behavior. Nonlinearity arises from the term $6uu_x$, which represents the interaction of the wave with itself. Dispersion, on the other hand, is represented by the terms u_{xxx} and $3\sigma^2 u_{yy}$, which describe how the wave spreads out over time and space.

The (2+1)-dimensional nature of KP The Kadomtsev-Petviashvili (KP) equation is actually a (2+1)-dimensional partial differential equation. It involves two spatial dimensions, x and y, and one temporal dimension, t. The equation describes the evolution of a function $u(x, y, t)$ in two spatial dimensions and time. The coefficient α determines the strength of the dispersion in the y direction. The (2+1)-dimensional nature of the KP equation allows for the propagation of waves and the interaction of these waves in two spatial dimensions, which gives rise to interesting and complex dynamics.

Laplace equation The Laplace equation is typically written in Cartesian coordinates as: $\Delta^2 u = 0$, where u is a scalar function of position (x, y, z), and Δ^2 represents the Laplace operator, also known as the Laplacian. The Laplacian is defined as the sum of the second partial derivatives of u with respect to each of the spatial coordinates:

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In simple terms, the Laplace equation states that the sum of the second derivative of u with respect to each coordinate is zero.

The non-linearity of Laplace Equation: The Laplacian operator (Δ^2) is a linear operator, and when applied to a function u, it produces a linear combination of the second partial derivatives of u. However, there are nonlinear equations that are closely related to the Laplace equation, such as the Poisson equation $(\Delta^2 u = f(u))$, where $f(u)$ is a nonlinear function of u. However, the Laplace equation itself is linear and describes linear phenomena, while nonlinear effects are introduced when additional terms or non-linear dependencies are incorporated into the equation.

The **Poisson equation** extends the Laplace equation by introducing a source term, which can be a function of the solution itself. In its general form, the Poisson equation is given by: $\Delta^2 u = f$, where u is a scalar function of position (x, y, z), Δ^2 represents the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

, and f is a known function that acts as the source or driving term. The Poisson equation arises in various scientific and engineering applications. To solve the Poisson equation, one typically needs to specify appropriate boundary conditions, which describe the behavior of the solution u on the boundaries of the domain. These conditions can include Dirichlet boundary conditions, where the values of u are specified on the boundary.

Burger equation is obtained as a result of combining nonlinear wave motion with linear diffusion and is the simplest model for analyzing combined effect of nonlinear advection and diffusion. For a given field and diffusion, the general form of Burger equation (also known as viscous Burger equation) in one space dimension is the dissipative system: $u_t + u.u_x = \epsilon u_x x$ where $\epsilon > 0$ is the constant of viscosity. This is the simplest PDE combining both non-linear propagation effects and diffusive effects. When the right term is removed from equation we obtain the hyperbolic PDE

$$u_t + u.u_x = v u_{xx}.$$

Non-linearity of Burger's Equation arises from the fact that the velocity u is multiplied by its own derivative $(\frac{\partial u}{\partial x})$, resulting in a quadratic term. This nonlinearity introduces complexities and interesting phenomena into the equation's behavior. One consequence of the nonlinearity is the formation of shock waves in the solution of Burger's equation. Shock waves are discontinuous changes in the solution that occur when a localized region of high velocity propagates through the fluid. Another effect of the nonlinearity is the generation of solitons, which are solitary wave solutions that maintain their shape while propagating at a constant speed. Solitons in Burger's equation are known as viscous or Korteweg-de Vries (KdV) solitons and arise due to the balance between the nonlinear convection and the diffusive effects. The nonlinearity of Burger's equation also contributes to the occurrence of other interesting phenomena, such as the development of instabilities, wave breaking, and the interaction of different wave components.

1.5. Exact Solutions of Partial Differential Equation

The fact of studying nonlinear partial differential equations is considered as a complicated and challenging endeavor. Compared with the variety of methods present in linear equations, the methods for nonlinear equations are limited to some specific categories. Due to the complexity in nature, there is no general method to solve the nonlinear partial differential equations. Thus, when dealing with a nonlinear partial differential equation, and the first stage is to linearize it or to avoid the nonlinear factors entirely. However, in investigating the behavior of the physical system, one often arrives across situations when the linearized model needs to be approximate. That is when the study of nonlinear models as such becomes imperative. In the nineteenth century, linear systems became the mathematical discipline and accomplished outstanding success throughout the sciences. On the other hand, due to the complex nature of nonlinear partial differential equations, they remained much harder to understand.

Consequently, It is interesting to obtain an exact invariant solution of nonlinear partial differential equations essential for exploring the sensitivity of physical phenomena with several important physical parameters described by constant and variable coefficients. These solutions of nonlinear partial differential equations provide a complete and precise description of the system being investigated, which can be used to extract valuable information about its properties. Thus, there has been more interest in finding the exact solutions of nonlinear equations during the last few decades. These solutions provide information about the various aspects of physical and nonlinear phenomena. Exact solutions can be used as models for physical experiments and benchmarks for testing numerical algorithms. These solutions can be a basis for perfecting and testing computer algebra software packages for solving differential equations. The explicit solutions for NLPDEs are rare, and the methods that generate the solutions are getting popular and increasingly sought.

So to find the exact solutions, there exist and developed several effective methods have been such as the Direct method [1], Bäcklund transformation [2], Inverse scattering transformation [3], tanh-sech method [4], modified (G'/G)-expansion method [5], extended tanh method [6], sine-cosine method [7], Hirota method [8], unified method [9, 10], homogeneous balance method [11], Jacobi elliptic function method [12], F-expansion method [13], variational iteration method [14], homotopy perturbation method [15].

Recently, many researchers have been providing much awareness on the application of constructing the solitons and solitary wave solutions of nonlinear PDEs [16–18], which occur in nonlinear phenomena. Consequently, several powerful methods have been determined to construct the solitons and solitary wave solutions, such as the direct algebraic method [19], auxiliary equation method [20], inverse scattering scheme [21], Bäcklund transform method [22], extended mapping method [23], and Lie symmetry method [24–26].

In this project, we study the invariance of certain non-linear partial differential equations in order to determine their similarity solutions. The methods we use for this purpose are as follows:

1.6. Unified Method:

The unified method is a useful method that has appeared in recent times for finding exact solutions of nonlinear partial differential equations (NLPDE's). New obtained exact solutions are different types of soliton wave properties along with trigonometric, hyperbolic, and rational functions solutions. The gained distinguished varieties of exact solutions contain vital applications in engineering and physics. With 3D, 2D, density, and contour graphical illustration, mathematical results explicitly exhibit the proposed algorithm's complete honesty and high performance. From the observation of the outcomes acquired, it is noticed that the unified method can generate essential effects in taking the exact solutions of (NLPDE's). The unified method is a useful method that has appeared in recent times for finding exact solutions of nonlinear partial differential equations (NLPDE's). New obtained exact solutions are different types of soliton wave properties along with trigonometric, hyperbolic, and rational functions solutions. The gained distinguished varieties of exact solutions contain vital applications in engineering and physics. With 3D, 2D, density, and contour graphical illustration, mathematical results explicitly exhibit the proposed algorithm's complete honesty and high performance.

From the observation of the outcomes acquired, it is noticed that the unified method can generate essential effects in taking the exact solutions of (NLPDE's).

1.6.1. Methodology. Gozukizil et al. [9] applied the unified method to construct divers solutions of the nonlinear partial differential equations in their study. The fundamental steps of the employed method are as follows:

Step 1: In general, the nonlinear partial differential equations with two independent variables t and x as follows

$$\mathcal{F}(\Theta_x\ ,\Theta_t\ ,\Theta_{xt}\ ,\Theta_{xx}\ ,\Theta_{xxx}\ ,\Theta_{xxt}\ ,\dots)=0, \tag{1.1}$$

where $\Theta(x,t)$ is an unknown function of x and t and $\mathcal{F}$ is a polynomial in Θ and its various derivatives in which the higher order derivative and nonlinear terms are both involve.

In order to find the solutions of given equation we commence with the following transformation

$$\Theta(x,t)=\Psi(X),\ \text{with}\ X=\alpha x-\beta t, \tag{1.2}$$

in which α_1 and α_2 are real parameters. With the procedure described above, we reduce equation (1.1) to the following ordinary differential equation(ODE)

$$\mathcal{G}(\Psi,\ \Psi',\ \Psi'',\ \Psi''',\ \dots)=0, \tag{1.3}$$

Set 2: Now we integrate equation (1.3) as many times as possible. Keep the integrating constant to zero.
We set up the trial solution of ODE equation (1.3) in the following form

$$\Psi(X)=K_0+\sum_{i=1}^{N}\Big[K_i\,f(X)^i+L_i\,f(X)^{-i}\Big]. \tag{1.4}$$

where $K_0,\ K_i,\ L_i\ (1\ \le\ i\ \le\ N)$ are the arbitrary coefficients and explicit invariant function $f(X)$ in equation (1.4) is satisfy the following Riccati differential equation

$$f'(X)=f^2(X)+M, \tag{1.5}$$

The nine solutions of the above equation (1.5) are given by in three cases:

Case 1: Hyperbolic function solutions (when M is negative)

$$\begin{cases} (i)\ f(X) &= \dfrac{\sqrt{-(P^2+Q^2)M} - P\sqrt{-M}\cosh(2\sqrt{-M}(X+\phi))}{P\sinh(2\sqrt{-M}(X+\phi))+Q}, \\ (ii)\ f(X) &= \dfrac{-\sqrt{-(P^2+Q^2)M} - P\sqrt{-M}\cosh(2\sqrt{-M}(X+\phi))}{P\sinh(2\sqrt{-M}(X+\phi))+Q}, \\ (iii)\ f(X) &= \sqrt{-M} + \dfrac{2P\sqrt{-M}}{P+\cosh(2\sqrt{-M}(X+\phi))-\sinh(2\sqrt{-M}(X+\phi))}, \\ (iv)\ f(X) &= -\sqrt{-M} + \dfrac{2P\sqrt{-M}}{P+\cosh(2\sqrt{-M}(X+\phi))-\sinh(2\sqrt{-M}(X+\phi))}. \end{cases} \tag{1.6}$$

Case 2: Trigonometric function solutions (when M is positive)

$$\begin{cases} (v)\ f(X) &= \dfrac{\sqrt{(P^2+Q^2)M} - P\sqrt{M}\cos(2\sqrt{M}(X+\phi))}{P\sin(2\sqrt{M}(X+\phi))+Q}, \\ (vi)\ f(X) &= \dfrac{-\sqrt{(P^2+Q^2)M} - P\sqrt{M}\cos(2\sqrt{M}(X+\phi))}{P\sin(2\sqrt{M}(X+\phi))+Q}, \\ (vii)\ f(X) &= i\sqrt{M} + \dfrac{2iP\sqrt{M}}{P+\cos(2\sqrt{M}(X+\phi))-i\sin(2\sqrt{M}(X+\phi))}, \\ (viii)\ f(X) &= -i\sqrt{M} + \dfrac{2iP\sqrt{M}}{P+\cos(2\sqrt{M}(X+\phi))-i\sinh(2\sqrt{M}(X+\phi))}. \end{cases} \tag{1.7}$$

Case 3: Rational function solutions (when $M = 0$)

$$(ix)\ f(X) = -\frac{1}{X+\phi}, \tag{1.8}$$

when $P \neq 0$, ϕ and Q are arbitrary parameters.

Set 3: We determine the positive integer N in equation (1.3) by taking into account the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (1.3).

Set 4: By using the equation (1.4) into equation (1.3) with the use of equation (1.5) and gathering the coefficients of $f^k(X)$, we are get a set of algebraic equations and then solving this system of algebraic equations, we obtain several sets of solutions.

Set 5: Substituting sets of solutions which is obtained in step 4 and using the general solutions of (1.5) in step 2, explicit solutions of (1.3) can be obtained immediately depending on the Value M.

1.7. Solitons

Solitons are fascinating and important solutions that arise in the context of nonlinear partial differential equations (PDEs). A soliton is a localized, self-reinforcing waveform that can maintain its shape and propagate over long distances without dissipating or spreading out. It behaves as a particle-like entity, preserving its energy, momentum, and other physical properties during its motion. Nonlinear PDEs are mathematical equations that describe a wide range of physical phenomena, including fluid dynamics, optics, quantum mechanics, and more. Solitons typically emerge as solutions to these equations when nonlinear effects are present.
One of the remarkable aspects of solitons is their stability. Unlike most waveforms, solitons can interact with other solitons and emerge unchanged, except for a phase shift or velocity change. This property is known as soliton stability or the ”integrability” of the underlying PDE. Integrable equations possess an infinite number of conserved quantities, which are instrumental in preserving the soliton's characteristics during interactions. Solitons have practical applications in various fields. In fiber optics, solitons are employed to transmit data over long distances without distortion. They are also relevant in understanding rogue waves in the ocean, where localized waves with exceptionally large amplitudes can form and persist. Solitons are also crucial in studying nonlinear phenomena in plasmas, condensed matter physics, and many other areas of science and engineering. In summary, solitons are special solutions of nonlinear PDEs that maintain their shape, propagate without dissipating, and interact with other solitons while preserving their properties. They have significant implications in various branches of science and technology, making them an exciting and rich topic of study.

1.7.1. Role of Solitons in (PDE's). In the context of (PDE's), solitons arise as solutions to certain nonlinear equations that exhibit a balance between dispersion (spreading out) and nonlinearity (self-interaction). They are characterized by their stability, robustness, and ability to maintain their shape and integrity over long distances.

(1) Wave Propagation: Solitons can propagate through a medium while retaining their shape and velocity.
(2) Nonlinear Dynamics: Solitons are nonlinear phenomena that emerge due to the interplay between nonlinearity and dispersion in PDEs. They provide insights into the rich dynamics and behavior of nonlinear systems and offer a means to understand complex phenomena.

(3) Stability Analysis: Solitons often correspond to stable solutions of PDEs. Stability analysis involves studying the behavior of small perturbations around solitonic solutions to determine their long-term behavior. This analysis provides valuable information about the stability or instability of the system.
(4) Integrability: Solitons are intimately connected to integrable systems. Integrable PDEs possess an infinite number of conserved quantities, making their solutions soliton-like. This property allows for exact analytical solutions, leading to a deeper understanding of the underlying dynamics.
(5) Information Transmission: Solitons have the unique property of preserving their shape during propagation, making them useful for information transmission. They are employed in fields like optical communications, where soliton pulses can propagate long distances without significant distortion.
(6) Mathematical Physics: The study of solitons has contributed to the development of mathematical physics, including advancements in nonlinear analysis, inverse scattering transform, and the theory of integrable systems. Solitons have been instrumental in exploring the connections between PDEs, symmetries, and conservation laws.

1.7.2. Applications of solitons: The unique properties of the solitons and stability make them valuable tools in various scientific disciplines, enabling advances in technology, communication, and fundamental research. One of the most significant applications of solitons is in fiber optics. Soliton pulses can propagate over long distances without significant distortion, making them ideal for high-speed data transmission.

(1) Nonlinear Optics: Solitons play a crucial role in nonlinear optics, where the interaction of light with nonlinear materials leads to fascinating phenomena. Soliton pulses can propagate through nonlinear media, maintaining their shape and stability.
(2) Water Waves: Solitons can occur in water waves, forming stable localized wave packets that maintain their shape and travel over long distances. These solitons are observed in various natural phenomena, such as tidal bores, tsunamis, and rogue waves. Understanding solitonic behavior in water waves helps in predicting and mitigating the impact of extreme wave events.

Chapter 2

Exact solitary waves solutions of the (1+1)- dimensional Lonngren Equation

Finding soliton solutions of nonlinear partial differential equation (NLPEs) plays an essential role in study of various fields of sciences, especially in physics. Recently various dominant methods have been offered for finding solitons solutions, for instance: double $(G/G, 1/G)$-expansion method, Baffle-type Vortex Generators, Unified Method etc.The objective of this work is to provide various multiwave kinks for (1+1)-dimensional longitudinal lonngren wave equation,

$$\frac{\partial^2}{\partial t^2}\left(u_{xx} - \alpha u + \beta u^2\right) + u_{xx} = 0$$

where α and β are constants. This equation describes the electrical signals in telegraph lines on the basis of the tunnel diode. It was used as an example to show the existence of strong connection between the G/G expansion method and the modified extended tanh method. Other uses include modelling of waves in shallow water such as beaches, lakes and rivers. It offers adequate general soliton solutions which are connected in diverse branches of scientific fields.

2.0.1. Verification of Unified method on Lonngren wave equation. In general, the non-linear partial differential equations with two independant variables x and t as follows:

$$P\left(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}\right) = 0, \tag{2.1}$$

where $u(x,t)$ is an unknown function of x and t and P is a polynomial in u and its various derivatives in which the higher order derivative and non-linear terms are both involved.In order to find the solutions of Lonngren wave equation we commence with the following transformation:

$$u(x,t) = u(\epsilon),$$

with

$$\epsilon = x - ct,$$

in which c is a real variable. With the procedure described above, we reduce equation (2.1) to the following ordinary differential equation (ODE).

$$F(U, U'', U'') = 0. \tag{2.2}$$

Now, we integrate equation (2.2), Keep the integration constant to zero.
We set up the trial solution, the solution (1.4)

$$u(\epsilon) = a_0 + \sum_{i}^{M} \left[a_i \phi' + b_i \phi^{-1} \right].$$

where $\phi = \phi(\epsilon)$ is a solution of riccati equation: $\phi' = \phi^2 + b$.

Now, we balance the linear terms of highest order with the non-linear term of highest degree in the new version of above equation to find M, while using (1.5)

2.0.2. Lonngren Equation and verification by unified method: In this section, numerous exact-soliton solutions for a non linear lonngren wave equation are addressed by using the unified method.

$$\frac{\partial^2}{\partial t^2} \left(u_{xx} - \alpha u + \beta u^2 \right) + u_{xx} = 0 \tag{2.3}$$

Partially differentiating (2.3), with respect to t.

$$u_{xxtt} - u_{tt} + u_{tt}^2 + uxx = 0 \tag{2.4}$$

We assume the following transformation for converting the Lonngren's Wave Equation to Ordinary Differential Equation.

$$u(x,t) = U(\epsilon),\ u_x = U'(\epsilon),\ u_t = -cU'(\epsilon),\ u_{xx} = U''(\epsilon) \tag{2.5}$$

$$u_{tt} = c^2 U''(\epsilon),\ u_{xxtt} = c^2 U''''(\epsilon),\ u_{tt}^2 = c^2 U^2(\epsilon)''. \tag{2.6}$$

Substituting (2.6) into (2.4), we have

$$c^2\, U''''(\epsilon) - \alpha c^2\, U''(\epsilon) + \beta c^2\, U''^2(\epsilon) + U''(\epsilon) = 0 \tag{2.7}$$

Integrating (2.7) twice and putting integration constant zero, we have

$$c^2 U''(\epsilon) + (1 - \alpha c^2) U(\epsilon) + \beta c^2 U^2 = 0 \tag{2.8}$$

2.1. SUMMARY OF THE UNIFIED METHOD

Implementing balancing principle on U'' and U^2 of the above ODE (2.7) to find the value M, we have $2M = M + 2$ implies $M = 2$

$$U(\epsilon) = a_0 + \frac{a_1}{\phi} + \frac{b_1}{\phi} + a_2\phi^2 + \frac{b_2}{\phi^2} \tag{2.9}$$

$$U^2(\epsilon) = \left[a_0 + a_1\phi + \frac{b_1}{\phi} + a_2\phi^2 + \frac{b_2}{\phi^2}\right]^2 \tag{2.10}$$

$$U'(\epsilon) = a_1\phi' - \phi' b_1/\phi^2 + 2a_2 \prime^2 \prime^1 - 2\phi' b_2/\phi^3 \tag{2.11}$$

$$U''(\epsilon) = a_1\phi'' + 2a_2\phi'\phi' + 2a_2\phi''\phi' - b_1\left[\frac{\phi''}{\phi^2} - 2\frac{\phi'\phi'}{\phi^3}\right] - 2b_2\left[\frac{\phi''}{\phi^3} - 3\frac{\phi'\phi'}{\phi^4}\right] \tag{2.12}$$

Substituting this in (2.12)

$$U''(\epsilon) = 2ba_1\phi + 8ba_2\phi^2 + 2a_1\phi^3 + 6a_2\phi^4 + 2b_2 + 2a_2b_2 + \frac{2bb_1}{\phi} + \frac{8bb_2}{\phi^2} + \frac{2b_1b_2}{\phi^3} + \frac{2b_2b_2}{\phi^4}$$

Here, we consider some sets of solutions to derive the exact solitary wave solutions of via the computational software *Mathematica*.

$$\begin{aligned}
2a_1bc^2 + a_1 - \alpha c^2 a_1 + 2\beta c^2 a_0 a_1 + 2\beta c^2 a_2 b_1 &= 0,\\
8a_2bc^2 + a_2 - \alpha c^2 a_2 + 2\beta c^2 a_0 a_2 + \beta c^2 a_1^2 &= 0,\\
2a_1c^2 + 2\beta c^2 a_1 a_2 &= 0,\\
6a_2c^2 + \beta c^2 a_2^2 &= 0,\\
2b_1bc^2 + b_1 - \alpha c^2 b_1 + 2\beta c^2 a_0 b_1 + 2\beta c^2 a_1 b_2 &= 0,\\
8b_2bc^2 + b_2 - \alpha c^2 b_2 + 2\beta c^2 a_0 b_2 + \beta c^2 a_1 b_1 &= 0,\\
6b_2b^2c^2 + \beta c^2 b_2^2 &= 0,\\
2a_2b^2c^2 + 2b_2c^2 + a_0 - \alpha c^2 a_0 + \beta c^2 a_0^2 + 2\beta c^2 a_1 b_1 + 2\beta c^2 a_2 b_2 &= 0.
\end{aligned}$$

2.1.1. Solution Set:
After getting solution sets of coefficients in mathematica:

$$U(\epsilon) = a_0 + a_1\phi + a_2\phi^2 + \frac{b_1}{\phi} + \frac{b_2}{\phi^2} \tag{2.13}$$

Set-1:

$$c = \pm\frac{1}{\sqrt{4b+m}}, b_1 = 0, b_2 = 0, a_1 = 0, a_0 = \frac{-6b}{n}, a_2 = \frac{-6}{n} \tag{2.14}$$

Case-1: $b < 0,$

$$\phi = \frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B} \tag{2.15}$$

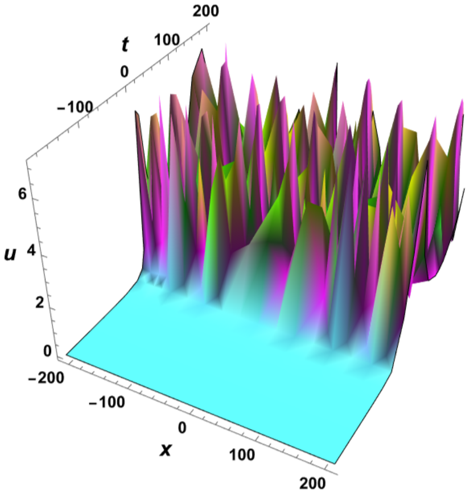


FIGURE 2.1. Plot3D of the Equation (2.17)

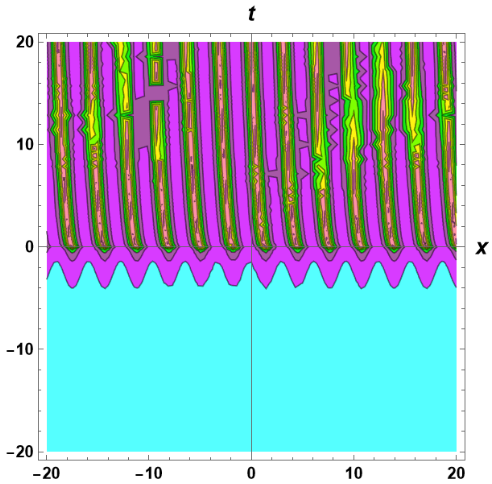


FIGURE 2.2. Contour Plot of the Equation (2.17)

Putting this and set-1 values in equation (2.13) to obtain $U(\epsilon)$

$$U(\epsilon) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0))+B}\right]^2 \tag{2.16}$$

$$u(x,t) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))}{A\sinh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))+B}\right]^2, \tag{2.17}$$

$$u(x,t) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(-A^2+B^2)b} + A\sqrt{-b}\cosh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))}{A\sinh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))+B}\right]^2, \tag{2.18}$$

$$u(x,t) = -\frac{6\sqrt{-b}}{n}\left[1+\frac{-2A}{A+\cosh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0)) - \sinh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))}\right]^2, - \frac{6b}{n}, \tag{2.19}$$

$$u(x,t) = -\frac{6\sqrt{-b}}{n}\left[1+\frac{2A}{A+\cosh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0)) + \sinh(2\sqrt{-b}(x\pm\frac{t}{\sqrt{4btm}}+\xi_0))}\right]^2 - \frac{6b}{n}. \tag{2.20}$$

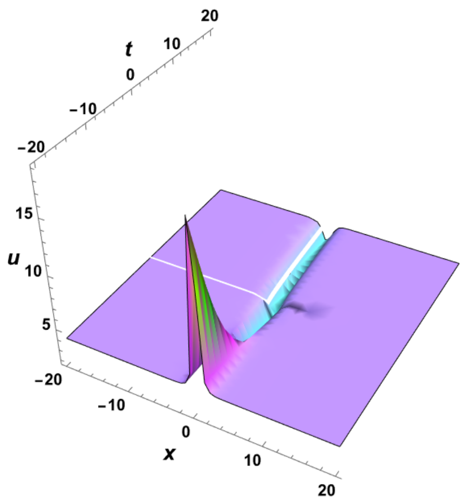


FIGURE 2.3. Plot3D of the Equation (2.23)

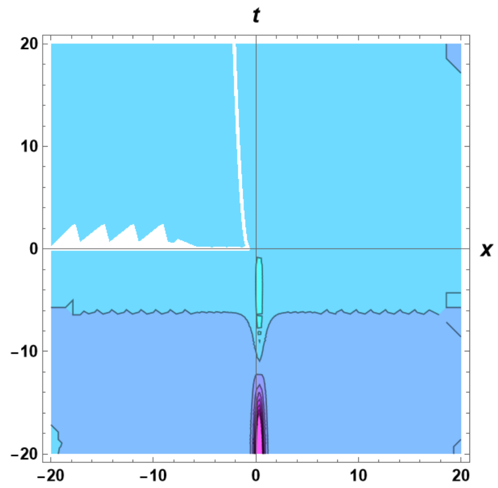


FIGURE 2.4. Contour Plot of the Equation (2.23)

Case-2: $b > 0$,

$$\phi = \frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{b}(\epsilon+\xi_0)) + B} \tag{2.21}$$

$$U(\epsilon) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{b}(\epsilon+\xi_0)) + B}\right]^2 \tag{2.22}$$

$$u(x,t) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.23}$$

$$u(x,t) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{\sqrt{(A^2+B^2)b} + A\sqrt{b}\cosh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.24}$$

$$u(x,t) = -\frac{6}{n}\left[\imath\sqrt{b} + \frac{-2A\imath\sqrt{b}}{A + \cos(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) - \imath\sin(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}\right]^2 - \frac{6b}{n}, \tag{2.25}$$

$$u(x,t) = -\frac{6}{n}\left[-\imath\sqrt{b} + \frac{2A\imath\sqrt{b}}{A + \cos(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + \imath\sin(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}\right]^2 - \frac{6b}{n}. \tag{2.26}$$

Case-3: $b = 0$,

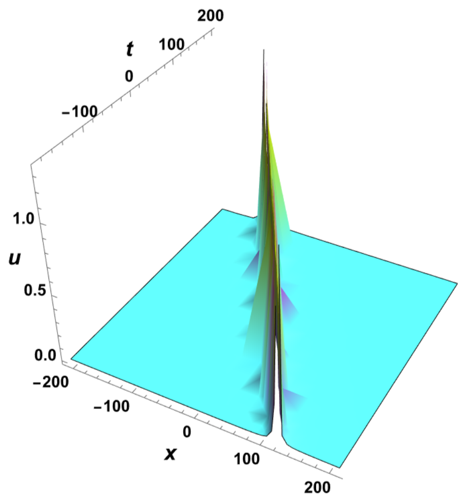


FIGURE 2.5. Plot3D of the Equation (2.29)

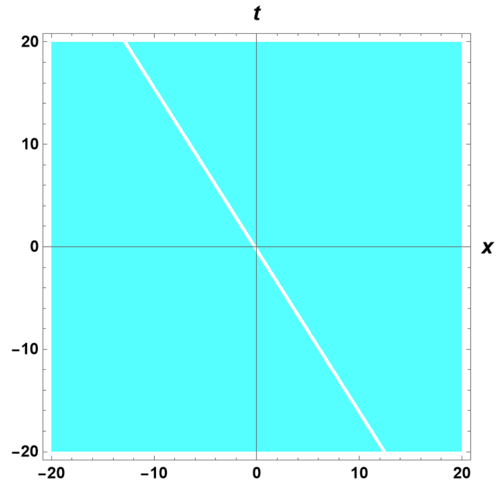


FIGURE 2.6. Contour Plot of the Equation (2.29)

$$U(\epsilon) = \frac{-6b}{n} - \frac{6}{n}\left[\frac{-1}{\epsilon + \xi_0}\right]^2 \tag{2.27}$$

$$u(x,t) = \frac{-6}{n}\left[\frac{-1}{x \pm \frac{1}{\sqrt{m}} + \xi_0}\right]^2 \tag{2.28}$$

$$u(x,t) = \frac{-6}{n(x \pm \frac{t}{\sqrt{m}} + \xi_0)^2}. \tag{2.29}$$

Set-2

$$c = \pm\frac{1}{\sqrt{4b+m}}, b_1 = 0, b_2 = 0, a_1 = 0, a_0 = \frac{-2b}{n}, a_2 = \frac{-6}{n} \tag{2.30}$$

Using the transformations as used above: Case-1: $b < 0$,

$$\phi = \frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon + \xi_0))}{A\sinh(2\sqrt{-b}(\epsilon + \xi_0)) + B} \tag{2.31}$$

Putting this and set-2 values in above equation to obtain $U(\epsilon)$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{-b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.32}$$

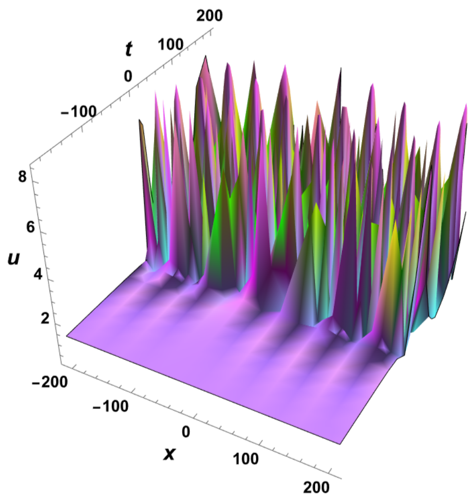


FIGURE 2.7. Plot3D of the Equation (2.32)

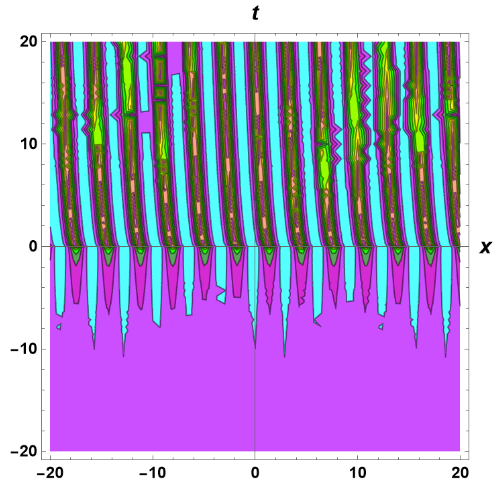


FIGURE 2.8. Contour Plot of the Equation (2.32)

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\frac{\sqrt{(-A^2+B^2)b} + A\sqrt{-b}\cosh(2\sqrt{-b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{-b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.33}$$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\sqrt{-b} + \frac{-2A\sqrt{-b}}{A + \cosh(\Phi) - \sinh(\Phi)}\right]^2, \tag{2.34}$$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\sqrt{-b} + \frac{2A\sqrt{-b}}{A + \cosh(\Phi) + \sinh(\Phi)}\right]^2. \tag{2.35}$$

Where $\Phi = 2\sqrt{-b}(x \pm \dfrac{t}{\sqrt{4btm}} + \xi_0)$.

Case-2: $b > 0$,

$$\phi = \frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon + \xi_0))}{A\sinh(2\sqrt{b}(\epsilon + \xi_0)) + B} \tag{2.36}$$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.37}$$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\frac{\sqrt{(A^2+B^2)b} + A\sqrt{b}\cosh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}{A\sinh(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + B}\right]^2, \tag{2.38}$$

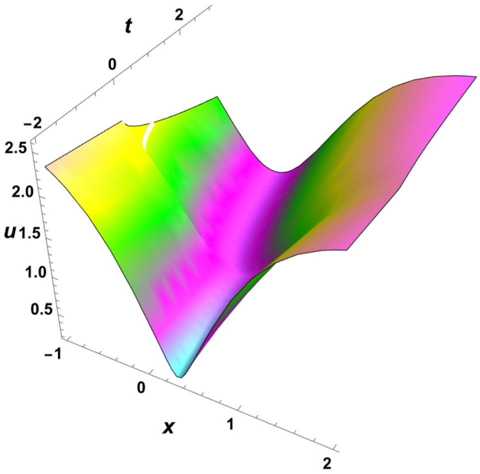


FIGURE 2.9. Plot3D of the Equation (2.37)

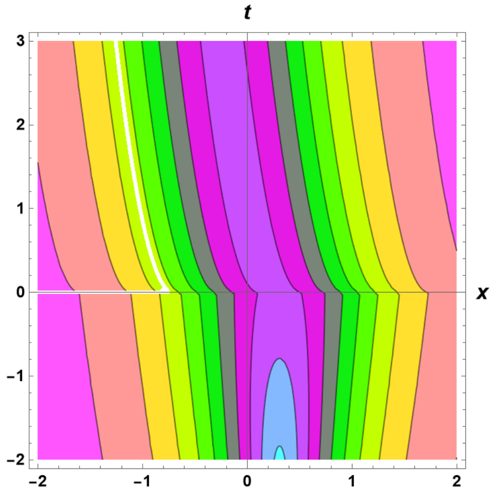


FIGURE 2.10. Contour Plot of the Equation (2.37)

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[\iota\sqrt{b} + \frac{-2A\iota\sqrt{b}}{A+\cos(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) - \iota\sin(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}\right]^2, \tag{2.39}$$

$$u(x,t) = \frac{-2b}{n} - \frac{6}{n}\left[-\iota\sqrt{b} + \frac{2A\iota\sqrt{b}}{A+\cos(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0)) + \iota\sin(2\sqrt{b}(x \pm \frac{t}{\sqrt{4btm}} + \xi_0))}\right]^2. \tag{2.40}$$

Case-3 : $b = 0$,

$$U(x,t) = \frac{-6}{n}\left[\frac{-1}{x \pm \frac{1}{\sqrt{m}} + \xi_0}\right]^2 \tag{2.41}$$

$$u(x,t) = \frac{-6}{n(x \pm \frac{t}{\sqrt{m}} + \xi_0)^2} \tag{2.42}$$

As it can be seen that unified method gives more than 49 solutions for the Lonngren wave equation when account the positiveness or negativeness of values.

2.2. RESULTS AND DISCUSSIONS:

In this chapter, we have done a study of solitons in the context of non linear partial differential equations solved using the Unified method. We looked at some various NLPDEs, into the Lonngren wave equation. Unified method has been used to solve these NLPDEs as it has been observed that it can generate essential effects in taking the exact solutions of NLPDEs. The solutions that arise in this context, solitons, are localized, self-reinforcing waveforms that can maintain their shape and propagate over long distances without dissipating or spreading out. They have applications in non linear optics and naturally occurring phenomena like water waves. We verified the Lonngren wave equation by using the Unified method and generated solution sets and illustrated the same with multiwave 3D plots and contour plots using Mathematica.

(1) Figure 1,2: Plots of multiwave solution in (2.17) are presented via n=3, A=1, B=1.2, b=1, $\epsilon = 0.2$, m=2.5, (1) multiwave 3D plot, (2) contour plot respectively.
(2) Figure 3,4: Plots of multiwave solution in (2.23) are presented via n=3, A=1, B=1.2, b=1,$\epsilon = 0.2$, m=2.5, (3) multiwave 3D plot, (4) contour plot respectively.
(3) Figure 5,6: Plots of multiwave solution in (2.29) are presented via n=3,$\epsilon = 0.2$, m=2.5, (5) multiwave 3D plot, (6) contour plot respectively.
(4) Figure 7,8: Plots of multiwave solution (2.32) are presented via n=3, A=1, B=1.2, b=1,$\epsilon = 0.2$, m=2.5, (7) multiwave 3D plot, (8) contour plot respectively.
(5) Figure 9,10: Plots of multiwave solution in (2.37) are presented via n=3, A=1, B=1.2, b=1,$\epsilon = 0.2$, m=2.5, (9) multiwave 3D plot, (10) contour plot respectively.

Chapter 3

Exact soliton solutions of the (1+1)-dimensional of the Burger's Equation

Burgers equation is a partial differential equation in fluid dynamics. It makes an attempt to explain the behaviour of viscous fluids under certain conditions. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics. Burgers (1948) first developed this equation so as to throw light on the turbulence described by the interaction of two opposite effects of convection and diffusion. In the (1.4) u represents the velocity of the fluid as function of time and position t and x respectively. $\frac{\partial u}{\partial t}$ represents change of velocity with respect to time t. $\frac{\partial u}{\partial x}$ represents the spatial gradient velocity. v is the small diffusivity.

The term uu_x expresses the shocking up effect that causes the waves to break up, and the term is the diffusion term. The Burgers equation is a nonlinear equation because of u_x the term in the advection portion of the equation. This equation helps in capturing the non linear effects of fluid dynamics. The presence of the diffusion term prevents the gradual distortion of the wave and its breaking by countering the nonlinearity. The result is a balance between the nonlinear advection term and the linear diffusion term much the same way as occurs in a real shock wave in the narrow region where the gradient is steep. It mimics the Navier Stokes equation of fluid motion. If we think of u as the velocity then the right hand side of the Burgers equation represents the momentum being advected by the deterministic component of the flow, while the right hand side depicts the diffusion through thermal fluctuations.

The Burgers equation combines convection (advection) and diffusion (viscosity) effects of fluid mechanics. It expresses the balance between the fluids tendency to flow (convection) with its tendency to spread out or diffuse due to its viscosity. It can give slight insights into aspects of turbulence especially in one dimensional circumstances.

3.0.1. Burger's Equation and verification by unified method:

$$u_t + u.u_x = nu_{xx}$$

We assume the following transformation to convert the given Burger's Equation into Ordinary Differential Equation.

$$u(x,t) = U(\epsilon) \tag{3.1}$$

Partially Differentiating (3.1) with respect to x, we get:

$$u_x = U'(\epsilon),$$
$$u_{xx} = U''(\epsilon),$$
$$u_t = -cU\prime(\epsilon) = -cu''.$$

Integrating the above equation required times and putting the integration constant to 0. Here $M = 1$ Now putting the values of ϕ' and ϕ'' using the Riccati equation,

$$\phi'(\epsilon) = \phi^2(\epsilon) + b, \tag{3.2}$$

$$U(\epsilon) = a_0 + a_1\phi + \frac{b_1}{\phi}, \tag{3.3}$$

$$U'(\epsilon) = a_1\phi' - \frac{b_1\phi'}{\phi^2}, \tag{3.4}$$

$$So, U'(\epsilon) = a_1(\phi^2 + b) - \frac{b_1}{\phi^2}(\phi^2 + b). \tag{3.5}$$

Differentiating Above Equation (3.5):

$$U''(\epsilon) = 2a_1\phi\phi' + \frac{2bb_1\phi'}{\phi^3}, \tag{3.6}$$

$$U''(\epsilon) = 2a_1\phi^3 + 2a_1b\phi + \frac{2bb_1}{\phi} + \frac{2b^2b_1}{\phi^3}. \tag{3.7}$$

Integrating the above equation (3.5) required times and putting the integration constant to 0. Putting these value above using riccati equation,we get:

$$\phi^4[a_1^2b - 2a_1bn] = 0,$$
$$\phi^5[a_0a_1 - a_1c] = 0,$$
$$\phi^6[a_1^2 - 2a_2n] = 0,$$
$$\phi^2[-b_1^2 - 2bb1n] = 0,$$
$$\phi[-a_0bb1 + bb_1c] = 0,$$
$$-bb_1^2 - 2b^2b_1n = 0,$$

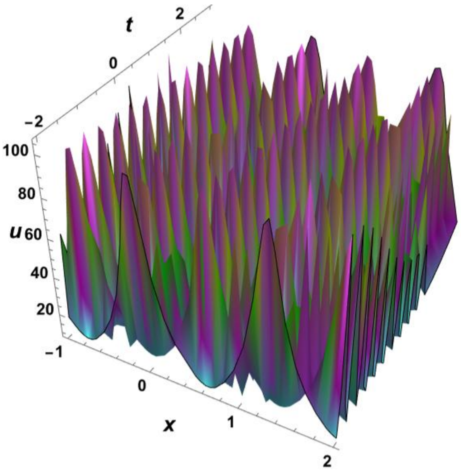


FIGURE 3.1. Plot3D of the Equation (3.11)

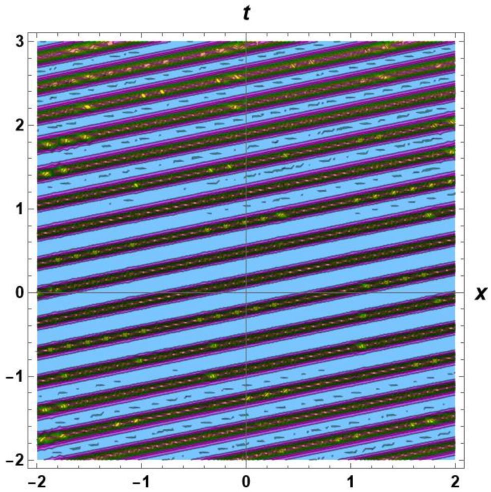


FIGURE 3.2. Contour Plot of the Equation (3.11)

$$\phi^3[a_0a_1b - a_0b_1 - a_bc + b_1c] = 0.$$

3.0.2. Solution Set: After getting solution sets of coefficients in mathematica:

$$U(\epsilon) = a_0 + a_1\phi + \frac{b_1}{\phi}. \tag{3.8}$$

Set-1: Here $a_0 = c$ and $a_1 = 2n$ and $b_1 = 0$.

Case-1: $b < 0$,

$$\phi = \frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B}. \tag{3.9}$$

$$U(\epsilon) = a_0 + a_1\phi + \frac{b_1}{\phi}. \tag{3.10}$$

$$u_1(x,t) = c + 2n\left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}\right], \tag{3.11}$$

$$u_2(x,t) = c + 2n\left[\frac{-\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}\right], \tag{3.12}$$

$$u_3(x,t) = c + 2n\left[\imath\sqrt{b} + \frac{-2A\imath\sqrt{b}}{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) - \imath\sin(2\sqrt{b}(\epsilon+\xi_0))}\right], \tag{3.13}$$

$$u_4(x,t) = c + 2n\left[-\imath\sqrt{b} + \frac{2A\imath\sqrt{b}}{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) + \imath\sin(2\sqrt{b}(\epsilon+\xi_0))}\right]. \tag{3.14}$$

Case-2: $b > 0$,

$$\phi = \frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{b}(\epsilon+\xi_0)) + B} \tag{3.15}$$

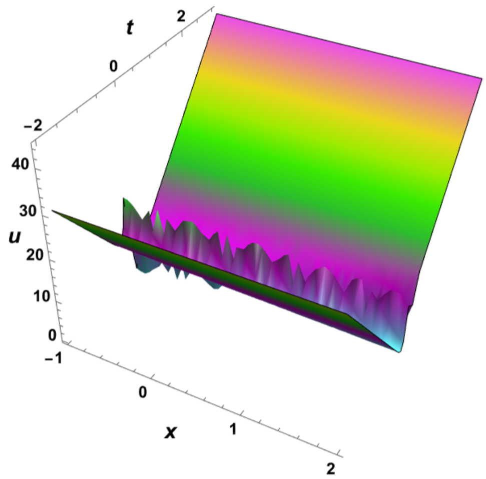


FIGURE 3.3. Plot3D of the Equation (3.16)

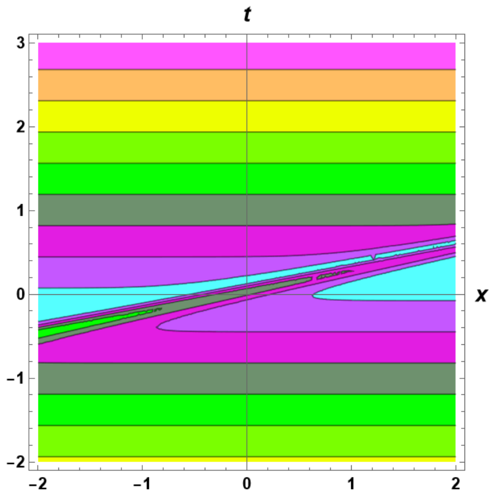


FIGURE 3.4. Contour Plot of the Equation (3.16)

$$u_5(x,t) = c + 2n \left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cos(2\sqrt{-b}(\epsilon+\xi_0))}{A\sin(2\sqrt{-b}(\epsilon+\xi_0)) + B} \right] \tag{3.16}$$

$$u_6(x,t) = c + 2n \left[\frac{-\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cos(2\sqrt{-b}(\epsilon+\xi_0))}{A\sin(2\sqrt{-b}(\epsilon+\xi_0)) + B} \right] \tag{3.17}$$

$$u_7(x,t) = c + 2n \left[\iota\sqrt{-b} + \frac{-2A\iota\sqrt{-b}}{A + \cos(2\sqrt{-b}(\epsilon+\xi_0)) - \iota\sin(2\sqrt{-b}(\epsilon+\xi_0))} \right] \tag{3.18}$$

$$u_8(x,t) = c + 2n \left[-\iota\sqrt{-b} + \frac{2A\iota\sqrt{-b}}{A + \cos(2\sqrt{-b}(\epsilon+\xi_0)) + \iota\sin(2\sqrt{-b}(\epsilon+\xi_0))} \right] \tag{3.19}$$

Case-3:$b = 0$,

$$u_9(x,t) = c + 2n \left[\frac{-1}{(\epsilon+\xi_0)} \right] \tag{3.20}$$

Set-2: Here $a_0 = c$ and $a_1 = 0$ and $b_1 = -2bn$

Case-1: $b < 0$,

$$\phi = \frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B} \tag{3.21}$$

$$U(\epsilon) = a_0 + a_1\phi + \frac{b_1}{\phi} \tag{3.22}$$

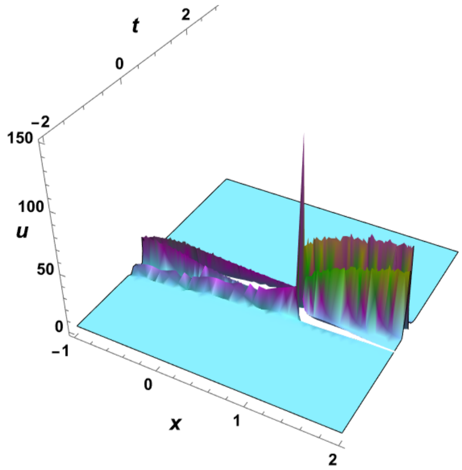


FIGURE 3.5. Plot3D of the Equation (3.23)

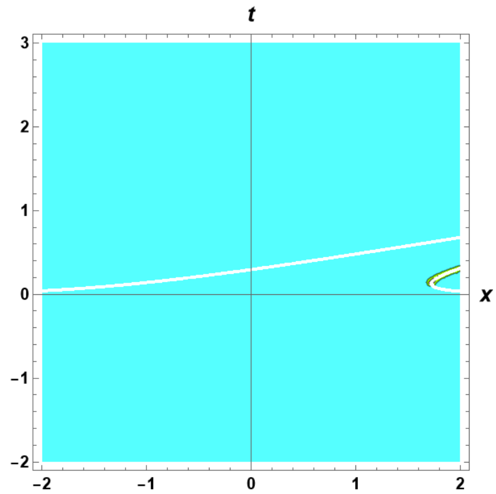


FIGURE 3.6. Contour Plot of the Equation (3.23)

$$u_{10}(x,t) = c - 2bn\left[\frac{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}\right] \tag{3.23}$$

$$u_{11}(x,t) = c - 2bn\left[\frac{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}{\sqrt{-(A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}\right] \tag{3.24}$$

$$u_{12}(x,t) = c - 2bn\left[\frac{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) - \iota\sin(2\sqrt{b}(\epsilon+\xi_0))}{\iota\sqrt{b}A + \iota\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sqrt{b}\sin(2\sqrt{b}(\epsilon+\xi_0)) - 2A\iota\sqrt{b}}\right] \tag{3.25}$$

$$u_{13}(x,t) = c - 2bn\left[\frac{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sin(2\sqrt{b}(\epsilon+\xi_0))}{-\iota\sqrt{b}A - \iota\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sqrt{b}\sin(2\sqrt{b}(\epsilon+\xi_0)) + 2A\iota\sqrt{b}}\right] \tag{3.26}$$

$$\tag{3.27}$$

Case-2: $b > 0$

$$\phi = \frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{b}(\epsilon+\xi_0)) + B} \tag{3.28}$$

$$u_{14}(x,t) = c - 2bn\left[\frac{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B}{-\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.29}$$

$$u_{15}(x,t) = c - 2bn\left[\frac{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B}{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.30}$$

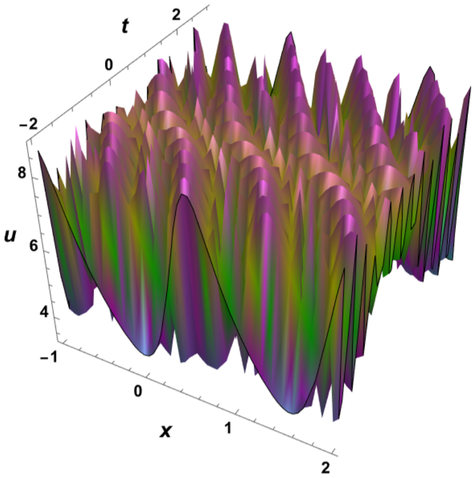


FIGURE 3.7. Plot3D of the Equation (3.29)

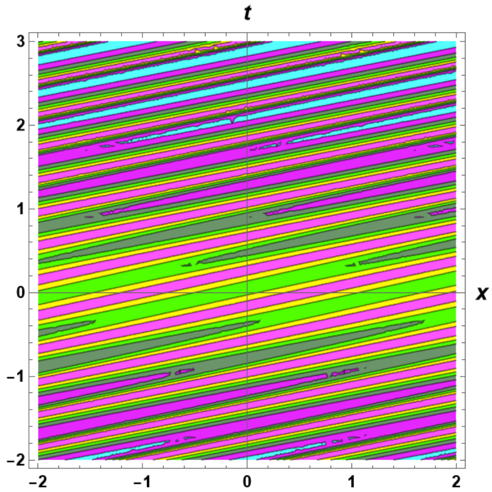


FIGURE 3.8. Contour Plot of the Equation (3.29)

$$u_{16}(x,t) = c - 2bn\left[\frac{A - \sinh(2\sqrt{-b}(\epsilon+\xi_0)) + \cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sqrt{-b} + \sqrt{b}\cosh(2\sqrt{-b}(\epsilon+\xi_0)) - \sqrt{-b}\sinh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.31}$$

$$u_{17}(x,t) = c - 2bn\left[\frac{A + \sinh(2\sqrt{-b}(\epsilon+\xi_0)) + \cosh(2\sqrt{-b}(\epsilon+\xi_0))}{-A\sqrt{-b} + \sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0)) - \sqrt{-b}\sinh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.32}$$

Case-3:$b = 0$,

$$u_{18}(x,t) = c + 2n\left[\frac{-1}{(\epsilon+\xi_0)}\right] \tag{3.33}$$

Set-3: Here $a_0 = c$ and $a_1 = 2n$ and $b_1 = -2bn$

Case-1: $b > 0$

$$\phi = \frac{\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B} \tag{3.34}$$

$$U(\epsilon) = a_0 + a_1\phi + \frac{b_1}{\phi} \tag{3.35}$$

Now, Converting Variable: $\epsilon = x - ct$.

$$\begin{aligned} u_{19}(x,t) &= c + 2n\left[\frac{\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}\right] \\ &- 2bn\left[\frac{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}{\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}\right] \end{aligned} \tag{3.36}$$

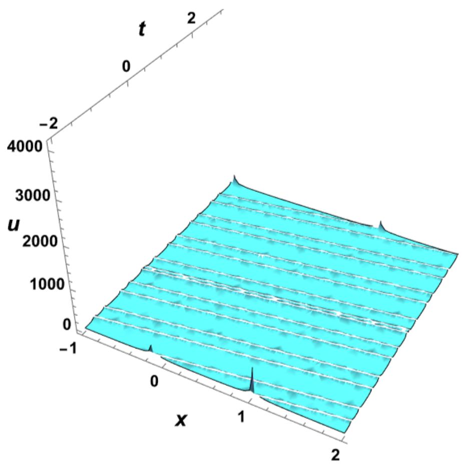


FIGURE 3.9. Plot3D of the Equation $U_{19}(x,t)$

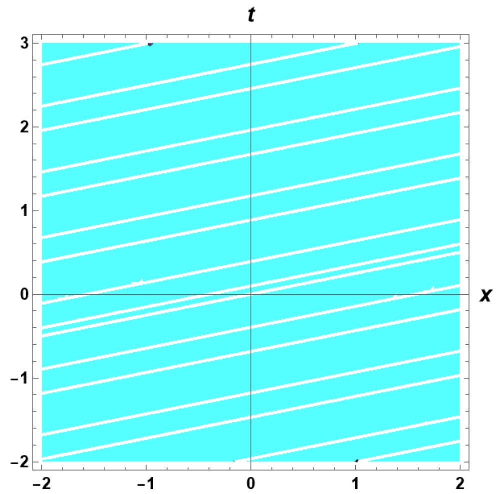


FIGURE 3.10. Contour Plot of the Equation $U_{19}(x,t)$

$$u_{20}(x,t) = c + 2n\left[\frac{-\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}\right] - 2bn\left[\frac{A\sin(2\sqrt{b}(\epsilon+\xi_0)) + B}{-\sqrt{(-A^2+B^2)b} - A\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0))}\right] \tag{3.37}$$

$$u_{21}(x,t) = c + 2n\left[\iota\sqrt{-b} + \frac{-2A\iota\sqrt{-b}}{A + \cos(2\sqrt{-b}(\epsilon+\xi_0)) - \iota\sin(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.38}$$

$$-2bn\left[\frac{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sin(2\sqrt{b}(\epsilon+\xi_0))}{-\iota\sqrt{b}A - \iota\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sqrt{b}\sin(2\sqrt{b}(\epsilon+\xi_0)) - 2A\iota\sqrt{b}}\right] \tag{3.39}$$

$$u_{22}(x,t) = c + 2n\left[-\iota\sqrt{-b} + \frac{2A\iota\sqrt{-b}}{A + \cos(2\sqrt{-b}(\epsilon+\xi_0)) + \iota\sin(2\sqrt{-b}(\epsilon+\xi_0))}\right] - 2b\left[\frac{A + \cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sin(2\sqrt{b}(\epsilon+\xi_0))}{-\iota\sqrt{b}A - \iota\sqrt{b}\cos(2\sqrt{b}(\epsilon+\xi_0)) + \iota\sqrt{b}\sin(2\sqrt{b}(\epsilon+\xi_0)) + 2A\iota\sqrt{b}}\right] \tag{3.40}$$

Case-2: $b > 0$.

$$\phi = \frac{\sqrt{(A^2+B^2)b} - A\sqrt{b}\cosh(2\sqrt{b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{b}(\epsilon+\xi_0)) + B} \tag{3.41}$$

$$u_{23}(x,t) = c + 2n\left[\frac{-\sqrt{(-A^2+B^2)b} - A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0)) + B}\right]$$

$$-2bn\left[\frac{A\sin(2\sqrt{-b}(\epsilon+\xi_0))+B}{\sqrt{(-A^2+B^2)b}-A\sqrt{b}\cos(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.42}$$

$$u_{24}(x,t)=c+2n\left[\frac{-\sqrt{(-A^2+B^2)b}-A\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{A\sinh(2\sqrt{-b}(\epsilon+\xi_0))+B}\right]$$

$$-2bn\left[\frac{A\sin(2\sqrt{-b}(\epsilon+\xi_0))+B}{-\sqrt{(-A^2+B^2)b}-A\sqrt{b}\cos(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.43}$$

$$u_{25}(x,t)=c+2n\left[\sqrt{-b}+\frac{-2A\sqrt{-b}}{A+\cos(2\sqrt{-b}(\epsilon+\xi_0))-\sin(2\sqrt{-b}(\epsilon+\xi_0))}\right]$$

$$-2bn\left[\frac{A+\sinh(2\sqrt{-b}(\epsilon+\xi_0))+\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{-A\sqrt{-b}+\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))-\sqrt{-b}\sinh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.44}$$

$$u_{26}(x,t)=c+2n\left[-\sqrt{-b}+\frac{-2A\sqrt{-b}}{A+\cos(2\sqrt{-b}(\epsilon+\xi_0))-\sin(2\sqrt{-b}(\epsilon+\xi_0))}\right]$$

$$-2bn\left[\frac{A+\sinh(2\sqrt{-b}(\epsilon+\xi_0))+\cosh(2\sqrt{-b}(\epsilon+\xi_0))}{-A\sqrt{-b}-\sqrt{-b}\cosh(2\sqrt{-b}(\epsilon+\xi_0))-\sqrt{-b}\sinh(2\sqrt{-b}(\epsilon+\xi_0))}\right] \tag{3.45}$$

$$\tag{3.46}$$

Case-3: $b=0$,

$$u_{27}(x,t)=c+2n\left[\frac{-1}{(\epsilon+\xi_0)}\right] \tag{3.47}$$

As it can be seen that unified method gives more than 49 solutions for the Burger's equation when account the positiveness or negativeness of values.

3.1. RESULTS AND DISCUSSIONS:

In this chapter, we have discussed the nature and physical significance of Burger's equation. To study the physical characteristics, Burger's equation is first solved by using the Unified method which generates different solution sets. Then some of the solutions obtained by different solution sets are plotted with multiwave 3D plots and contour plots using Mathematica.

(1) Figure 11,12 : Plots of multiwave solution in (3.11) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.

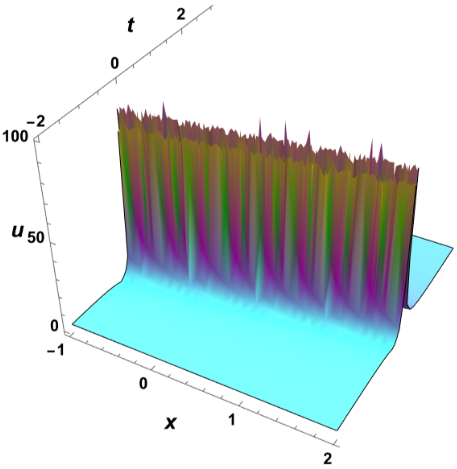


FIGURE 3.11. Plot3D of the Equation (3.47)

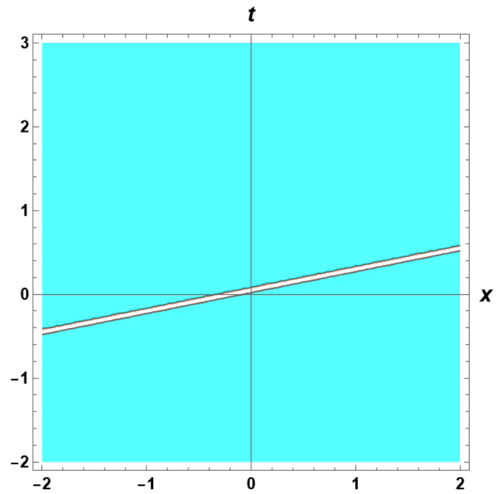


FIGURE 3.12. Contour Plot of the Equation (3.47)

(2) Figure 13,14 : Plots of multiwave solution in (3.16) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.
(3) Figure 15,16 : Plots of multiwave solution in (3.23) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.
(4) Figure 17,18 : Plots of multiwave solution in (3.29) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.
(5) Figure 19,20 : Plots of multiwave solution in (3.36) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.
(6) Figure 21,22 : Plots of multiwave solution in (3.47) are presented via n = 3, A = 1, B = 1.2, b = 5 ,$\epsilon = 0.2$,c = 4 ,(1) multiwave 3D plot,(2) contour plot respectively.

Chapter 4

Exact closed-form solutions of (1+1)-dimensional longitudinal wave equation

The objective of this work is to provide various soliton solutions for (1+1)-dimensional longitudinal wave equation in magneto electro-elastic circular (MEEC) rod,

$$u_{tt} - r^2 u_{xx} - \left(\frac{r^2}{2}u^2 + n u_{tt}\right)_{xx} = 0, \tag{4.1}$$

Where r is the wave velocity and n is the displacement parameter A travelling wave pulse known as a soliton is the result of specific nonlinear partial differential equations. Due to its exceptional stability characteristics, this particular wave may be used in numerous significant applications. The study of wave propagation in one-dimensional systems has been fundamental in understanding the behaviour of physical phenomena across various domains. Particularly, the investigation of longitudinal waves in rods has been of great interest due to its relevance in materials science, seismology, and other areas of physics. In recent years, the integration of theoretical concepts from Conformal Field Theory (CFT) has enriched our understanding of wave behaviour, introducing intriguing elements such as the Modified Effective Central Charge (MECC).In the context of a (1+1)-dimensional longitudinal wave equation for a rod with Modified Effective Central Charge (MECC), we can describe the wave propagation in the rod using a wave equation. The wave equation typically used for longitudinal waves is a variation of the 1D wave equation. We take into account wave propagation in a long MEE circular rod. Z is along the rod direction, or the direction in which waves propagate, in the cylindrical coordinate system. The following assumptions are made to aid our study:

- The rod's cross-section remains plain both before and after the deformation.

- The rod's lateral surface has axial symmetry.

Wave propagation in magneto-electro elastic(MEE) media has been studied by numerous researchers as a result of the expanding applications of MEE structures in various engineering domains (such as actuators, sensors, etc.) over time .Using a modified $\exp(-\omega(\epsilon))$-expansion function approach, Baskonus et al. identified the hyperbolic function and complex hyperbolic function solutions of the nonlinear longitudinal wave equation (LWE) in a MEE circular rod. In their study of numerical solitary wave solutions, Xue et al. used the dispersion caused by the transverse Poisson's effect in a MEE circular rod to derive the nonlinear LWE. The ansatz, modified (G'/G)-expansion, and functional variable techniques are just a few of the innovative analytical solutions of the LWE in a MEE circular rod that have been researched .Recently, Seadawy used the extended trial equation method to find the soliton and other types of solutions of nonlinear LWE in a MEE circular rod. The combination of piezoelectric $BaTiO_3$ and piezomagnetic $CoFe_2O_4$ with various values of the volume fraction (vf) of piezoelectric $BaTiO_3$ is the bodily meaning of nonlinear LWE in MEE circular rod. The rod's radius is given as $r = 0.05$ m. By comprehensively understanding the wave propagation in magneto-electro-elastic circular rods, this research aims to contribute valuable insights to the design and optimization of devices and systems that rely on these unique materials. The results of this study have the potential to advance the development of innovative applications in fields ranging from aerospace engineering to medical devices. In the subsequent sections, we will detail the theoretical framework, mathematical derivations, numerical simulations, and experimental validations to provide a holistic perspective on the behaviour of longitudinal waves in magneto-electro-elastic circular rods. Ultimately, this research strives to enhance our knowledge of these materials and facilitate their broader utilization in emerging technologies.

4.1. APPLICATION OF THE (1+1)-DIMENSIONAL LONGITUDINAL WAVE EQUATION

In this section, numerous exact-soliton solutions for a nonlinear system (4.1) are addressed by using the unified method which are discussed in the above section 2.

Applying the wave transformation

$$u(x,t) = U(X), \text{ with } X = \alpha_1 x - \alpha_2 t, \tag{4.2}$$

in the (1+1)-dimensional longitudinal wave equation (4.1), we have

$$r^2(U'^2+(1+U)U'')+\alpha_2^2 nU^{(4)}-\frac{\alpha_2^2}{\alpha_1^2}U''=0. \tag{4.3}$$

Integrating equation (4.3), two times with respect to X and neglecting the integration constant, we obtained

$$\alpha_1^2\alpha_2^2 nU''+\frac{1}{2}\alpha_1^2 r^2 U^2+(\alpha_1^2 r^2-\alpha_2^2)U=0. \tag{4.4}$$

4.2. Summary of the Unified Method and Its Application

Implementing balancing principle on U'' and U^2 of the above ODE (4.4) to find the value M, we have $2M=M+2$ implies $M=2$. Consequently, equation (1.4) transform as

$$U(X)=K_0+K_1 f(X)+K_2 f(X)^2+\frac{L_1}{f(X)}+\frac{L_2}{f(X)^2}. \tag{4.5}$$

Substituting equation (4.5) into (4.4) with the Riccati equation (1.5) and succeeding the main steps of the employed method, that gives several kind of solution sets. Here, we consider some sets of solutions to derive the exact solitary wave solutions of (4.1) via the computational software *Mathematica*.

4.2.1. Solution Set: Set-1:

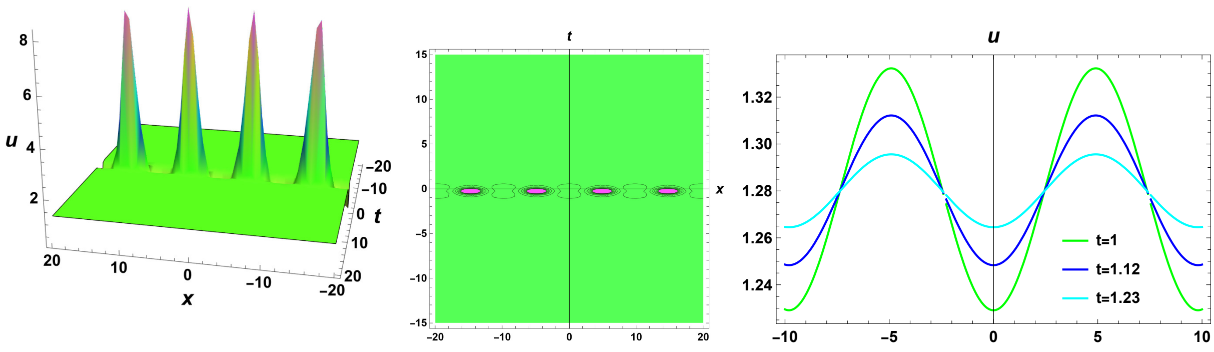
$$K_0=-\frac{4\alpha_2^2 Mn}{r^2},\ K_1=0,\ K_2=-\frac{12\alpha_2^2 n}{r^2},\ L_1=0,\ L_2=0,\ \alpha_1=\frac{\alpha_2}{\sqrt{4\alpha_2^2 Mn+r^2}}, \tag{4.6}$$

Making use of equation (4.5), and equation (4.6), the desired solution of the (1+1)-dimensional longitudinal wave equation obtained in the following cases,

Case-1: Hyperbolic function solutions (when $M<0$)

$$u_{11}(x,t)=-\frac{4\alpha_2^2\, n}{r^2}\left(M+\frac{3\left(\sqrt{-(\beta^2+\gamma^2)M}-\beta\sqrt{-M}\cos\left(2\sqrt{M}\,\Xi_1\right)\right)^2}{\left(\beta\sinh\left(2\sqrt{-M}\,\Xi_1\right)+\gamma\right)^2}\right), \tag{4.7}$$

$$u_{12}(x,t)=-\frac{4\alpha_2^2\, n}{r^2}\left(M+\frac{3\left(\sqrt{-(\beta^2+\gamma^2)M}+\beta\sqrt{-M}\cos\left(2\sqrt{M}\,\Xi_1\right)\right)^2}{\left(\beta\sinh\left(2\sqrt{-M}\,\Xi_1\right)+\gamma\right)^2}\right). \tag{4.8}$$

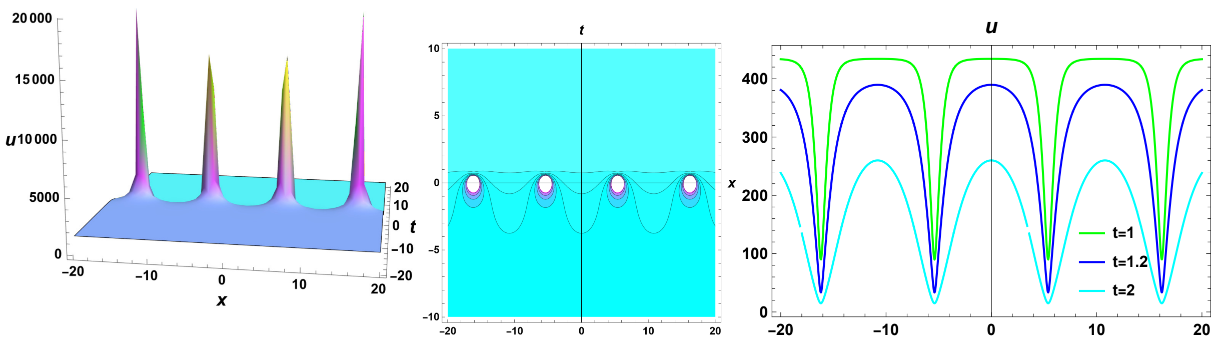


(A) (B) (C)

FIGURE 4.1. Evolutionary profile dynamics of the solution (4.7).

$$u_{13}(x,t) = \frac{4\alpha_2^2 Mn}{r^2}\left(3\left(\frac{2\beta}{\exp\left(-2\sqrt{-M}\left(\frac{\alpha_2 x}{\sqrt{4\alpha_2^2 Mn + r^2}} - \alpha_2 t + \phi\right)\right) + \beta} + 1\right)^2 - 1\right), \tag{4.9}$$

$$u_{14}(x,t) = \frac{4\alpha_2^2 Mn}{r^2}\left(3\left(\frac{2\beta}{\exp\left(-2\sqrt{-M}\left(\frac{\alpha_2 x}{\sqrt{4\alpha_2^2 Mn + r^2}} - \alpha_2 t + \phi\right)\right) + \beta} - 1\right)^2 - 1\right). \tag{4.10}$$



(A) (B) (C)

FIGURE 4.2. Evolutionary profile dynamics of the solution (4.8).

Case-2: Trigonometric function solutions (when $M > 0$)

$$u_{15}(x,t) = -\frac{4\alpha_2^2 n}{r^2}\left(\frac{3\left(\sqrt{M(\beta^2+\gamma^2)} - \sqrt{M}\beta\cos\left(2\sqrt{M}\,\Xi_1\right)\right)^2}{\left(\beta\sin\left(2\sqrt{M}\,\Xi_1\right)+\gamma\right)^2} + M\right), \tag{4.11}$$

$$u_{16}(x,t) = -\frac{4\alpha_2^2 n}{r^2}\left(\frac{3\left(\sqrt{M}\beta\cos\left(2\sqrt{M}\,\Xi_1\right) + \sqrt{M(\beta^2+\gamma^2)}\right)^2}{\left(\beta\sin\left(2\sqrt{M}\,\Xi_1\right)+\gamma\right)^2} + M\right), \tag{4.12}$$

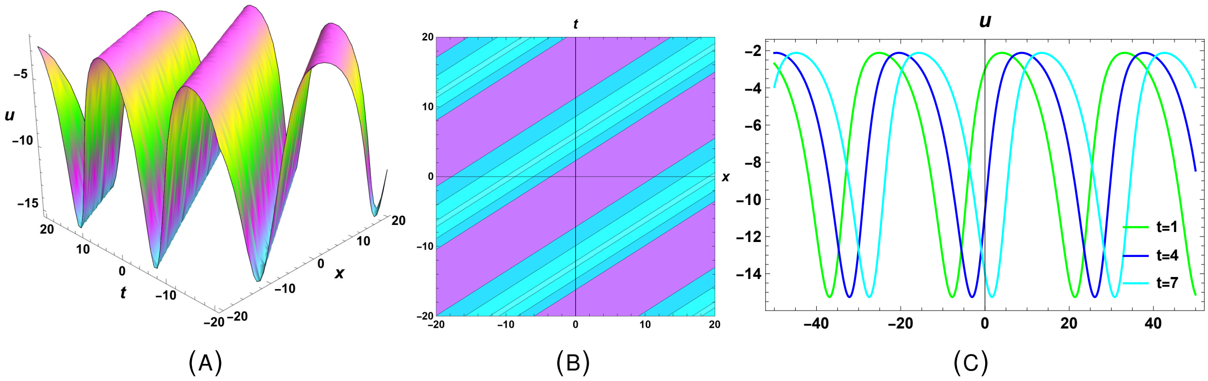


FIGURE 4.3. Evolutionary profile dynamics of the solution (4.11).

$$u_{17}(x,t) = \frac{4\alpha_2^2 Mn}{r^2}\left(-1+3\left(1+\frac{2\beta}{\beta+e^{-2i\sqrt{M}\left(\frac{\alpha_2 x}{\sqrt{4\alpha_2^2 Mn+r^2}}-\alpha_2 t+\phi\right)}}\right)^2\right), \tag{4.13}$$

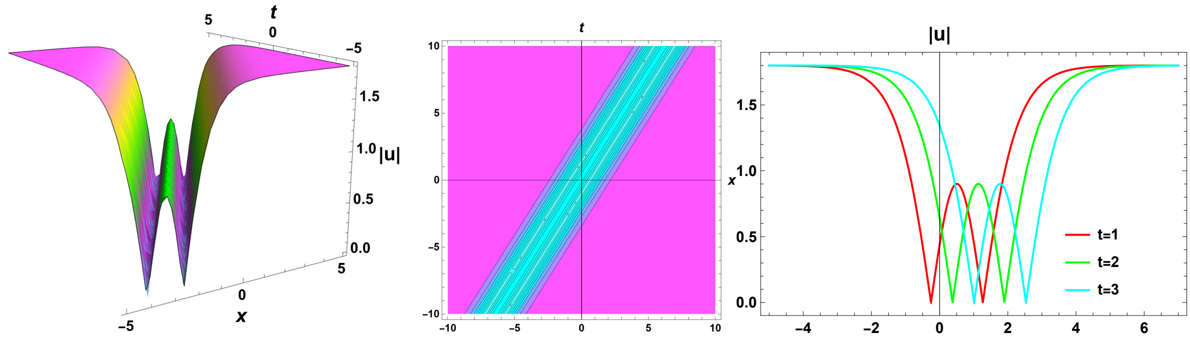
$$u_{18}(x,t) = \frac{4\alpha_2^2 Mn}{r^2}\left(-1+3\left(1-\frac{2\beta}{\beta+e^{-2i\sqrt{M}\left(\frac{\alpha_2 x}{\sqrt{4\alpha_2^2 Mn+r^2}}-\alpha_2 t+\phi\right)}}\right)^2\right). \tag{4.14}$$

Case-3: Rational function solutions (when $M = 0$)

$$u_{19}(x,t) = \frac{4\alpha_2^2 n}{r^2}\left(-\frac{3}{\left(\frac{\alpha_2 x}{\sqrt{4\alpha_2^2 Mn+r^2}}-\alpha_2 t+\phi\right)^2} - M\right). \tag{4.15}$$

where $\Xi_1 = \dfrac{\alpha_2\, x}{\sqrt{4\alpha_2^2 Mn + r^2}} - \alpha_2\, t + \phi$.

Set-2:



(A) (B) (C)

FIGURE 4.4. Evolutionary profile dynamics of the solution (4.14).

$$K_0 = -\frac{12\alpha_2^2 M n}{r^2},\ K_1 = 0,\ K_2 = 0,\ L_1 = 0,\ L_2 = -\frac{12\alpha_2^2 M^2 n}{r^2},\ \alpha_1 = \frac{\alpha_2}{\sqrt{r^2 - 4\alpha_2^2 M n}}. \tag{4.16}$$

Making use of equation (4.5), and equation (4.16), the desired solution of the (1+1)-dimensional longitudinal wave equation obtained in the following cases,

Case-1: Hyperbolic function solutions (when $M < 0$)

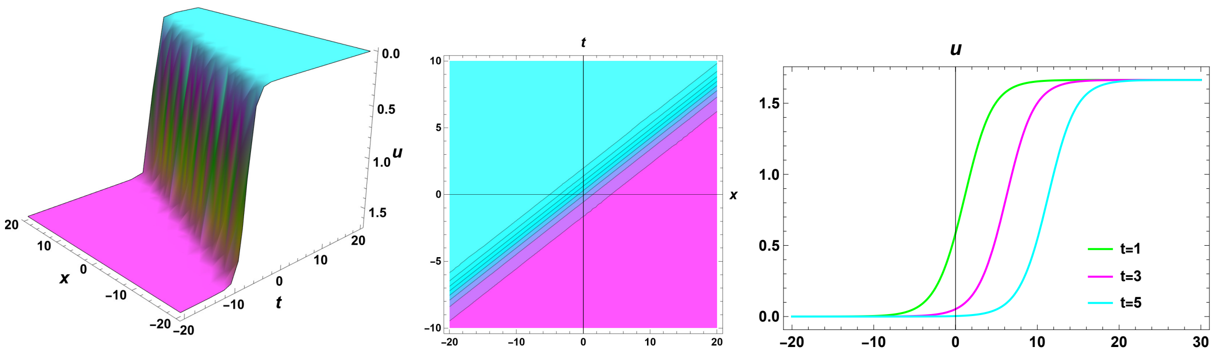
$$u_{21}(x,t) = -\frac{12\alpha_2^2 M n}{r^2}\left(\frac{M\left(\beta\sinh\left(2\sqrt{-M}\,\Xi_2\right)+\gamma\right)^2}{\left(\sqrt{-M\left(\beta^2+\gamma^2\right)}-\sqrt{-M}\beta\cos\left(2\sqrt{M}\,\Xi_2\right)\right)^2}+1\right), \tag{4.17}$$

$$u_{22}(x,t) = \frac{12\alpha_2^2 M n}{r^2}\left(-1-\frac{M\left(\beta\sinh\left(2\sqrt{-M}\,\Xi_2\right)+\gamma\right)^2}{\sqrt{-M\left(\beta^2+\gamma^2\right)}-\left(\sqrt{-M}\beta\cos\left(2\sqrt{M}\,\Xi_2\right)\right)^2}\right), \tag{4.18}$$

$$u_{23}(x,t) = \frac{12\alpha_2^2 M n}{r^2}\left(\left(\frac{2\beta}{\exp\left(-2\sqrt{-M}\left(\frac{\alpha_2 x}{\sqrt{r^2-4\alpha_2^2 M n}}-\alpha_2 t+\phi\right)\right)+\beta}+1\right)^{-2}-1\right), \tag{4.19}$$

$$u_{24}(x,t) = \frac{48\alpha_2^2 Mn\beta e^{2\sqrt{-M}\left(\frac{\alpha_2 x}{\sqrt{r^2-4\alpha_2^2 Mn}}-\alpha_2 t+\phi\right)}}{r^2\left(\beta e^{2\sqrt{-M}\left(\frac{\alpha_2 x}{\sqrt{r^2-4\alpha_2^2 Mn}}-\alpha_2 t+\phi\right)}-1\right)^2}. \tag{4.20}$$

Case-2: Trigonometric function solutions (when $M > 0$)



(A) (B) (C)

FIGURE 4.5. Evolutionary profile dynamics of the solution (4.19).

$$u_{25}(x,t) = -\frac{12\alpha_2^2 Mn}{r^2}\left(\frac{M\left(\beta\sin\left(2\sqrt{M}\,\Xi_2\right)+Q\right)^2}{\left(\sqrt{M\left(\beta^2+\gamma^2\right)}-\sqrt{M}\beta\cos\left(2\sqrt{M}\,\Xi_2\right)\right)^2}+1\right) \tag{4.21}$$

$$u_{26}(x,t) = -\frac{12\alpha_2^2 Mn}{r^2}\left(\frac{M\left(\beta\sin\left(2\sqrt{M}\,\Xi_2\right)+\gamma\right)^2}{\left(\sqrt{M}\beta\cos\left(2\sqrt{M}\,\Xi_2\right)+\sqrt{M\left(\beta^2+\gamma^2\right)}\right)^2}+1\right), \tag{4.22}$$

$$u_{27}(x,t) = -\frac{12\alpha_2^2 Mn}{r^2}\left(1+\left(\frac{2\beta}{\beta+e^{-2i\sqrt{M}\left(\frac{\alpha_2 x}{\sqrt{r^2-4\alpha_2^2 Mn}}-\alpha_2 t+\phi\right)}}+i\right)^{-2}\right), \tag{4.23}$$

$$u_{28}(x,t) = \frac{12\alpha_2^2 Mn}{r^2}\left(\left(1+\frac{2\beta}{\sin\left(2\sqrt{M}\,\Xi_2\right)-\cos\left(2\sqrt{M}\,\Xi_2\right)-\beta}\right)^{-2}-1\right). \tag{4.24}$$

Case-3: Rational function solutions (when $M = 0$)

$$u_{29}(x,t) = -\frac{12\alpha_2^2 Mn}{r^2}\left(M\left(\frac{\alpha_2 x}{\sqrt{r^2 - 4\alpha_2^2 Mn}} - \alpha_2 t + \phi\right)^2 + 1\right). \tag{4.25}$$

where $\Xi_2 = \left(\frac{\alpha_2 x}{\sqrt{r^2 - 4\alpha_2^2 Mn}} - \alpha_2 t + \phi\right)$.

4.3. Results and Discussion

In this section, we discuss the behavior and physical characteristics of some obtained solutions graphically using the modern Mathematical tool Mathematica. These solutions were obtained by using a powerful and effective unified method. The newly reported solutions show the different graphical behavior of the (1+1)-dimensions longitudinal wave equation. These solutions involve exact soliton solutions, fractional solutions, kink-wave soliton solutions, traveling wave solutions, singular-soliton solutions, periodic wave solutions, particular solutions, and w-shaped wave soliton solutions. A graph is an illustrated description of explicit solutions and is naturally drawn for comparison purposes.

Figure 4.1: Evolutionary profile dynamics of the solution (4.7) at $\alpha_2 = 2,\ r = 5,\ n = 1,\ M = -1,\ \beta = 0.1,\ \gamma = 0.2$, and $\phi = 0.22$ under the range space $-20 \le x \le 20,\ -20 \le t \le 20$ and figure (c) shows wave propagation with different value of $t = \{1,\ 1.12,\ 1.23\}$.

Figure 4.2: Evolutionary profile dynamics of the solution (4.8) at $\alpha_2 = 5,\ r = 11,\ n = 0.9,\ M = -5,\ \beta = 1,\ \gamma = 2$, and $\phi = 0.3$ under the range space $-20 \le x \le 20,\ -20 \le t \le 20$ and figure (c) shows wave propagation with different value of $t = \{1.2,\ 1.2,\ 2\}$.

Figure 4.3: Evolutionary profile dynamics of the solution(4.11) at $\alpha_2 = 1.2,\ r = 1.1,\ n = 11,\ M = 0.02,\ \beta = 0.22,\ \gamma = 0.5,\ \phi = 0.011$ under the range space $-20 \le x \le 20,\ -20 \le t \le 20$ and figure (c) shows wave propagation with different values of $t = \{1,\ 4,\ 7\}$

Figure 4.4: Evolutionary profile dynamics of the solution (4.14) at $\alpha_2 = 1.2,\ r = 2,\ n = 3,\ M = 0.3,\ \beta = 0.22,\ \phi = 0.2$ under the range space $-20 \le x \le 20,\ -20 \le t \le 20$ and figure (c) shows wave propagation with different values of $t = \{1.2,\ 1.22,\ 1.25\}$.

Figure 4.5: Evolutionary profile dynamics of the solution (4.19) at $\alpha_2 = 2,\ r = 2,\ n = 1.3,\ M = -0.12,\ \beta = 0.2,\ \phi = 1.2$ under the range space $-5 \le x \le 5,\ -5 \le t \le 5$ and figure (c) shows wave propagation with different values of $t = \{1,\ 2,\ 3\}$.

Concluding remarks

Most of the partial differential equations do not lead to a general solution, however, A powerful and effective unified method is used to reduce the given nonlinear equation to obtain differential equations. Such equation yield a family of the solitary waves, traveling waves and group invariant solutions of governing equation under consideration. Chapter 2, 3 and Chapter 3 utilized the unified method to attain the abundant exact solitary wave solutions and rational solutions to the the (1+1)-dimension Lonngren Equation, the (1+1)-dimensional Burger's Equation, the (1+1)-dimensional longitudinal wave equation. Our analytic results, obtained in this article are entirely novel and not yet published somewhere else. Furthermore, the method used in this chapter is a straightforward, effective, and productive technique in seeking the exact solitary wave solutions for many NLPDEs. We provided several interesting 3D, 2D-graphics, and respective density plots expressing the dynamical behavior of achieved solutions of the equations which give some structural information about how the behavior solutions are generated. Besides this, among the solutions, generalized rational solutions imply that these exact solitary wave solutions have rich local structures. Moreover, these solitary wave solutions will be useful to study analytically other nonlinear partial differential equations in plasma physics, nonlinear dynamics, materials physics, mathematical physics, applied sciences, and engineering. Consequently, our results have been verified with the aid of the symbolic computation via *Mathematica* by putting them back into the original equation.

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