SRI VENKATESWARA INTERNSHIP PROGRAM
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(SRI-VIPRA)

SRI-VIPRA

Project Report of 2024 : SVP-2446

"Exploring Some Ma-Minda functions in the field of Geometric Function Theory"

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SRIVIPRA PROJECT 2024

Title: Exploring Some Ma-Minda functions in the field of Geometric Function Theory.

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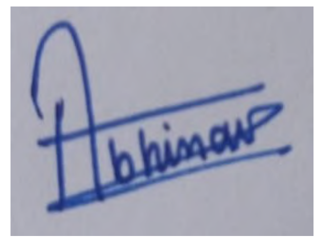
1		Abhinav	1722025	B.Sc(Hons.) Mathematics	
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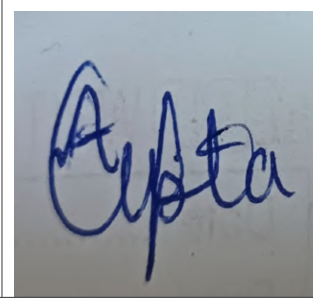
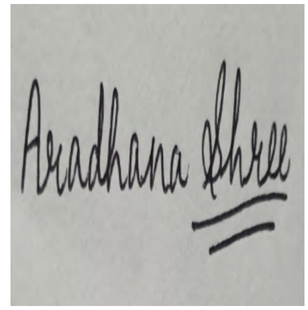
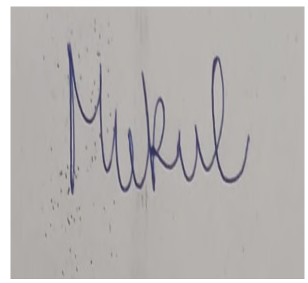
3		Akshali Gupta	17220902	B.Sc(Hons.) Mathematics	

4		Aradhana Shree	1722062	B.Sc(Hons.) Mathematics	
5		Mukul Gupta	1722063	B.Sc(Hons.) Mathematics	
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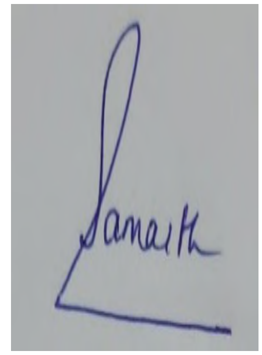
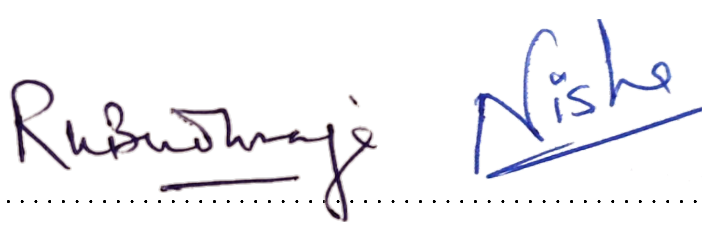
7		Samarth Dhingra	1722009	B.Sc(Hons.) Mathematics	



Signature of Mentors: ....

Certificate of Originality

This is to certify that the aforementioned students from Sri Venkateswara College have participated in the summer project SVP-2446 titled **"Exploring Some Ma-Minda Functions in the Field of Geometric Function Theory."** The participants have carried out the research project work under our guidance and supervision from July 1, 2024, to September 30, 2024. The work is original and conducted in a hybrid mode.

Signature of Mentors: ....

Acknowledgements

We are profoundly grateful to Sri Venkateswara College, University of Delhi, for providing us with the invaluable opportunity to undertake this research project. The endeavor would not have been possible without the guidance, support and encouragement of those to whom we express our heartfelt thanks.First and foremost, we would like to express our deepest gratitude to our supervisors and mentors, Prof. R.K. Budhraja and Dr. Nisha Bohra. Their insightful feedback and continuous encouragement challenged us to think critically and significantly improved the quality of our work. We are immensely grateful for their unwavering support and dedication, which has been instrumental in the successful completion of this project.

Contents

Chapter 1

Introduction

1.1 Basics of Complex Number

The Complex Plane

A complex number is represented by $a + ib$, where a and b are real, and i is the imaginary unit defined by $i^2 = -1$. For this complex number, a is termed the real part, while b is termed the imaginary part. The complex plane, also referred to as the Argand plane and symbolized by $\mathbb{C}$, is the space correlated with these numbers. A point $z = x + iy$ can be expressed as (x, y) and converted into polar coordinates as:

$$(x, y) = (r\cos\theta, r\sin\theta) \quad \text{where} \quad r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

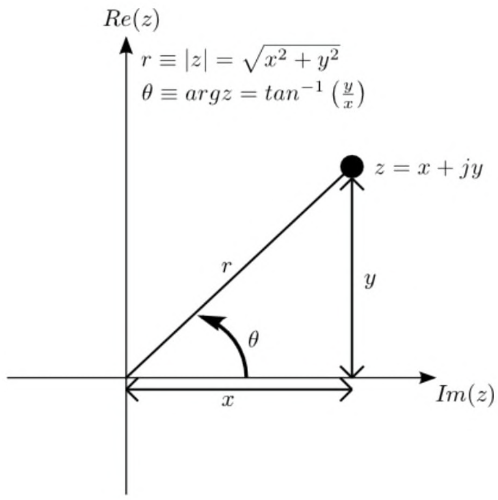


Figure 1: Representations of a complex number in the complex plane.

The non-negative real number r, also denoted by $|z|$, is referred to as the absolute value or modulus of z, while θ is known as the principal argument of z. By employing Euler's formula, which asserts that $e^{ix} = \cos x + i\sin x$ for $x \in \mathbb{C}$, we derive several equivalent expressions for a complex number:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

We broaden the framework $\mathbb{C}$ of complex numbers by incorporating the symbol ∞ to denote the point at infinity, or the ideal point. This modification turns the complex plane, along with the point at infinity, into the extended complex plane, symbolized by:

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Basic Algebraic Properties

Just as with real numbers, the properties of addition and multiplication also apply to complex numbers. Below, we outline some fundamental algebraic properties, demonstrating a few. The commutative laws state:

$$(1) \quad z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 z_2 = z_2 z_1$$

Also, the associative laws are:

$$(2) \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

These follow naturally from the principles of addition and multiplication for both complex and real numbers. The distributive law is similarly aligned:

$$(3) \quad (3) \quad z(z_1 + z_2) = z z_1 + z z_2$$

Example

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1$$

Due to the commutative property of multiplication, $iy = yi$ is valid. Therefore, one can express $z = x + yi$ as $z = x + iy$. Additionally, because of the associative properties, expressions like a sum $z_1 + z_2 + z_3$ or a product can be written without the need for parentheses, similar to real numbers.,The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers are equivalent in the complex number system. Specifically,

$$(4) \quad z + 0 = z \quad \text{and} \quad z \cdot 1 = z$$

for all complex numbers z. Moreover, 0 and 1 are uniquely the only complex numbers possessing these characteristics.

Inverse Properties

Each complex number $z = (x, y)$ has an additive inverse $-z = (-x, -y)$, which fulfills the condition $z + (-z) = 0$. For any complex number $z = (x, y)$ that is not zero, there is a multiplicative inverse satisfying $zz^{-1} = 1$. To determine this inverse, we look for real numbers u and v such that,

$$(x, y)(u, v) = (1, 0)$$

The only solution is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

Note that the inverse z^{-1} is undefined when $z = 0$.,

Shapes in the Complex Plane

In classical analytic geometry, the equation of a locus is given by a relationship between x and y. In the complex plane, it is expressed using z and $\bar{z}$.

Line

A straight line in the complex plane can be given by a parametric equation:

$$z = c + dt$$

where c and d are complex numbers and $d \neq 0$, while the parameter t runs through all real values.

Circle

The equation of a circle centered at a point $b \in \mathbb{C}$ with radius r is given by:

$$|z - b| = r$$

In algebraic form it can be written as

$$(z - b)(\bar{z} - \bar{b}) = r^2$$

The inside of the circle is described by the inequality $|z - b| < r$.

Parabola

The equation of a parabola with focus at $f_0 \in \mathbb{C}$ is given by:

$$|z - f_0| = \mathrm{Re}\left\{\frac{(z - c) \cdot \mathrm{i}\bar{d}}{|d|}\right\}$$

where the points $c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$ give the directrix $z = c + dt$ of the parabola, while the parameter t runs through all real values.

Ellipse

The general equation of an ellipse having foci b_1 and b_2 in $\mathbb{C}$ with the length of the semi-major axis $a \in \mathbb{R}$ is:

$$|z - b_1| + |z - b_2| = 2a$$

The centre of the ellipse is

$$b = \frac{b_1 + b_2}{2}$$

where $|c_1 - c_2| < 2b$.

Hyperbola

The general equation of a hyperbola having foci c_1 and c_2 in $\mathbb{C}$ with the length of the semi-major axis $b \in \mathbb{R}$ is:

$$||z - b_1| - |z - b_2|| = 2a$$

The centre of the hyperbola is

$$b = \frac{b_1 + b_2}{2}$$

where $|c_1 - c_2| > 2b$.

1.2 Topology of Complex Plane

The usual topology for the complex plane $\mathbb{C}$ is the topology induced by the standard metric defined as $d(z_1, z_2) = |z_1 - z_2|$. This document presents key concepts in the topology of complex functions.

Graphs

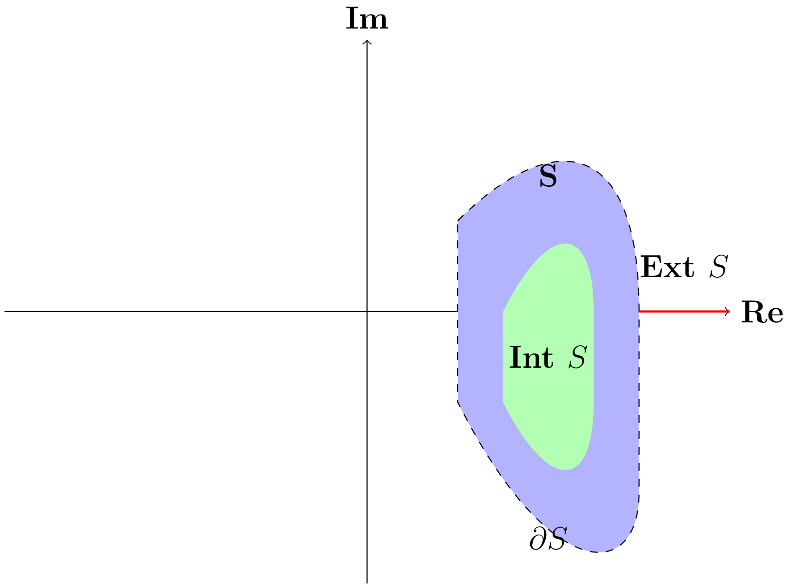


Figure 1: Int S, Ext S, and ∂S

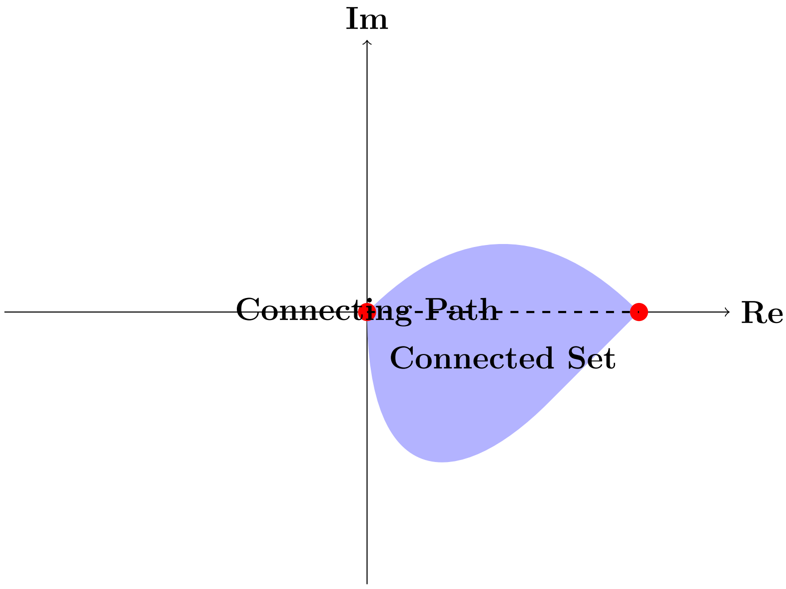


Figure 2: Connected Set

Definitions

1. **Neighborhood:** An ϵ-neighborhood of a complex number z_0 is given by:

$$B_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

2. **Interior Point:** A point z_0 is an interior point of a set $S \subseteq \mathbb{C}$ if $\exists\ \epsilon > 0$ such that $B_\epsilon(z_0) \subseteq S$.

3. **Exterior Point:** A point z_0 is an exterior point of S if $\exists\ \epsilon > 0$ such $that B_\epsilon(z_0) \cap S = \emptyset$.

4. **Boundary Point:** A point z_0 is a boundary point of S if every neighborhood $B_\epsilon(z_0)$ contains points in S and its complement S^c.

5. **Open Set:** A non-empty set $S \subseteq \mathbb{C}$ is open if Int $S = S$.

6. **Closed Set:** A non-empty set $S \subseteq \mathbb{C}$ is closed if it contains all its boundary points.

7. **Connected Set:** A non-empty set $S \subseteq \mathbb{C}$ is connected if any pair of points can be joined by a polygonal line that lies entirely in S.

8. **Domain:** A non-empty open set that is connected is called a domain.

1.3 Univalent Function

Let $\mathbb{D}$ be a non-empty open and connected set in the complex plane $\mathbb{C}$.

A function $f : \mathbb{D} \to \mathbb{C}$ is called *univalent* on $\mathbb{D}$ (or schlicht or one-to-one) if

$$f(z_1) = f(z_2) \implies z_1 = z_2$$

for all $z_1, z_2 \in \mathbb{D}$.

Example 1 - Rational Function:

$$f(z) = \frac{z}{1-z}$$

Proof: We need to check if

$$f(z_1) = f(z_2) \implies z_1 = z_2 :$$

$$\frac{z_1}{1-z_1} = \frac{z_2}{1-z_2}$$

Cross-multiplying gives:

$$z_1(1-z_2) = z_2(1-z_1)$$

This simplifies to:

$$z_1 - z_1z_2 = z_2 - z_1z_2$$

Rearranging gives:

$$z_1 = z_2$$

Hence, $f(z) = \frac{z}{1-z}$ is a univalent function.

Example 2 - Linear Function:

$$f(z) = az + b$$

Proof:
Assume $f(z_1) = f(z_2)$:

$$az_1 + b = az_2 + b$$

This simplifies to:

$$az_1 = az_2 \implies z_1 = z_2 \text{ (if } a \neq 0)$$

Hence, $f(z) = az + b$ is a univalent function.

We'll see additional examples with the help of the graphs.

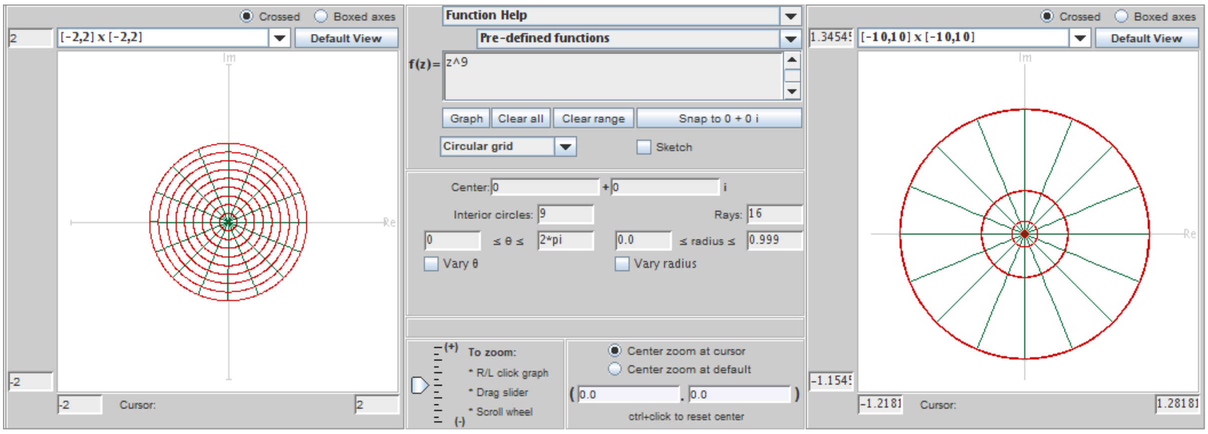


Figure 2: z^9

This function is univalent in $\mathbb{D}$

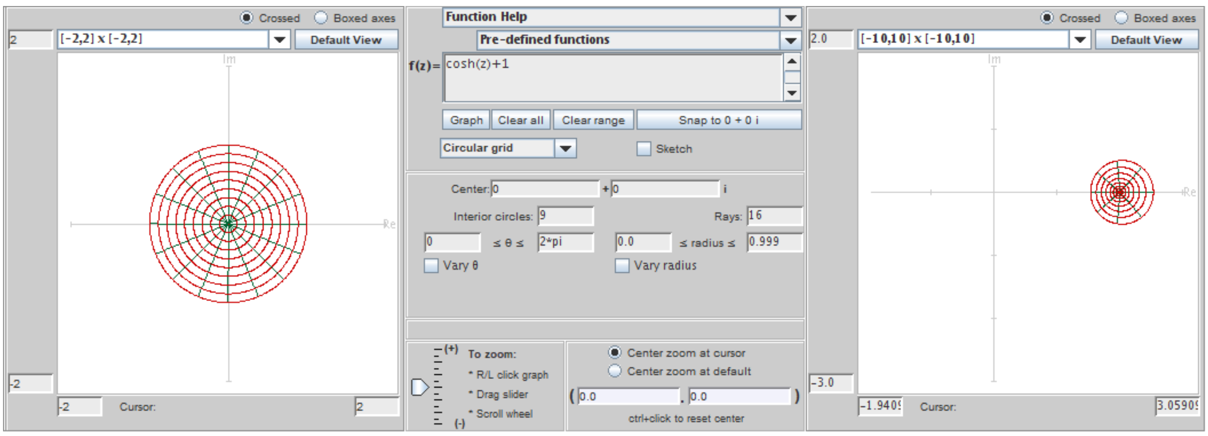


Figure 3: $\cosh(z) + 1$

This function is univalent in $\mathbb{D}$

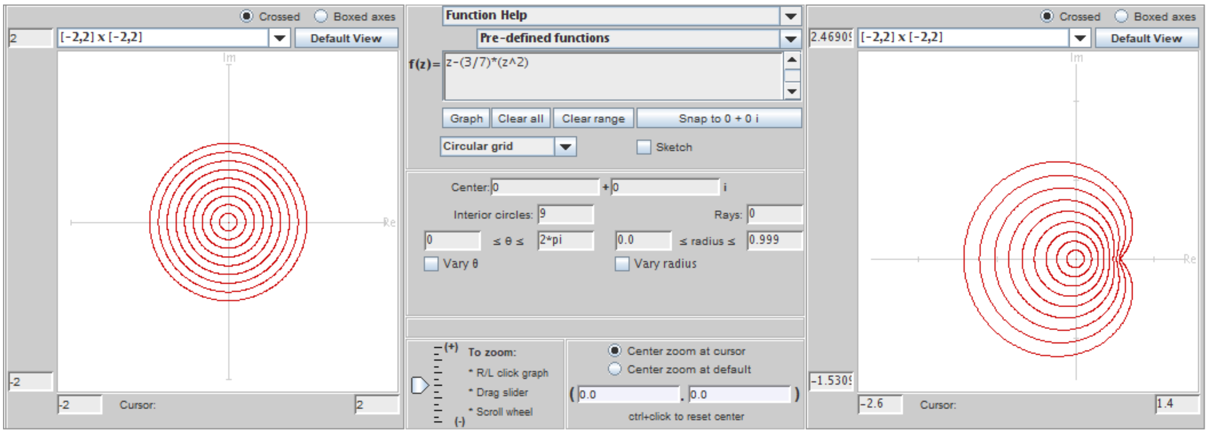


Figure 4: $z - \frac{3}{7}z^2$

This function is univalent in $\mathbb{D}$

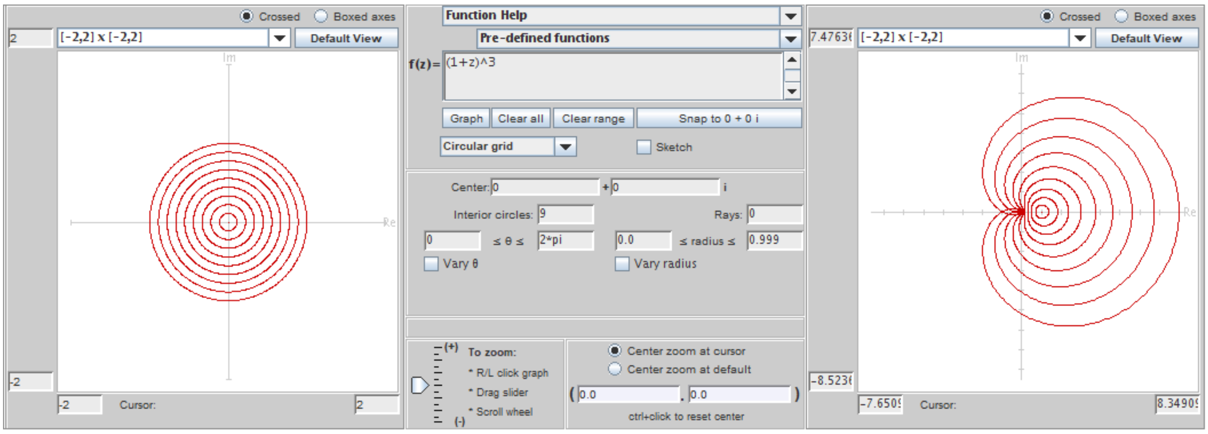


Figure 5: $(1 + z)^3$

This function is not univalent in $\mathbb{D}$ as the image curve has a point of self-intersection.

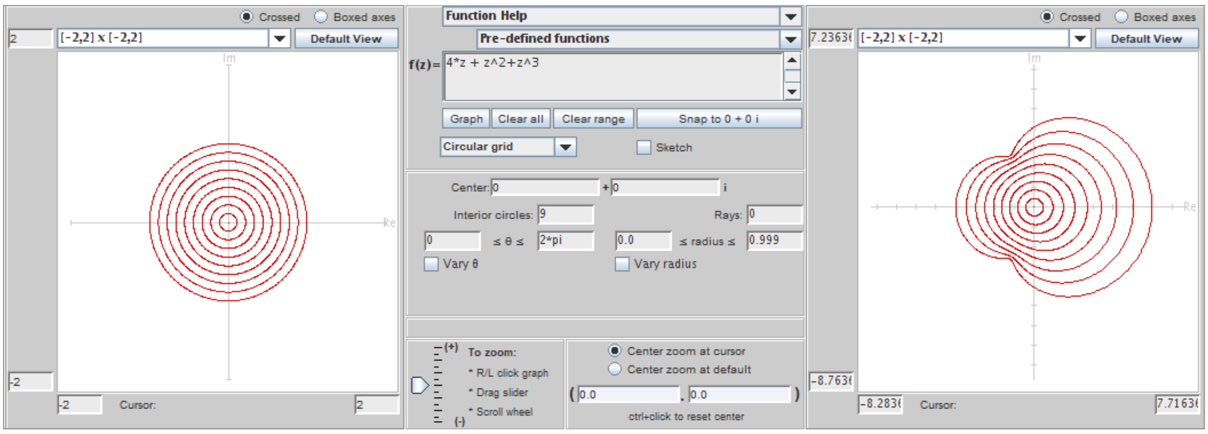


Figure 6: $4z + z^2 + z^3$

This function is univalent in $\mathbb{D}$

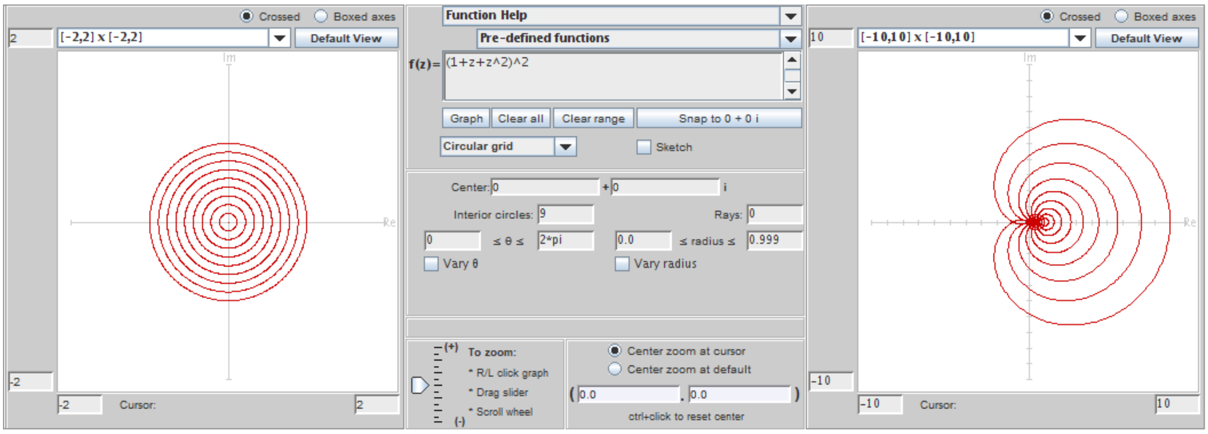


Figure 7: $(1+z+z^2)^2$

This function is not univalent in $\mathbb{D}$ as the image curve has a point of self-intersection.

Problem

Show that $f(z) = z + a_2 z^2$ is univalent in $\mathbb{D}$ if and only if $|a_2| \leq \frac{1}{2}$.

Proof

Suppose $f(z) = z + a_2 z^2$ is univalent in $\mathbb{D}$,then

$$f(z_1) = f(z_2) \implies z_1 = z_2$$

$$\begin{aligned} f(z_1) = f(z_2) &\iff z_1 + a_2 z_1^2 = z_2 + a_2 z_2^2 \\ &\iff (z_1 - z_2) + a_2(z_1 - z_2)(z_1 + z_2) = 0 \\ &\iff (z_1 - z_2)(1 + a_2(z_1 + z_2)) = 0. \end{aligned}$$

That is,

$$f(z_1) = f(z_2) \iff (z_1 - z_2)(1 + a_2(z_1 + z_2)) = 0 \quad .$$

Case 1: $z_1 = z_2$, which is trivial.
Case 2: $\quad 1 + a_2(z_1 + z_2) = 0 \quad \implies$

$$a_2 = \frac{-1}{z_1 + z_2}.$$

Now, note that

$$-2 \leq z_1 + z_2 \leq 2 \quad (\text{since } z_1, z_2 \in \mathbb{D}).$$

Thus,

$$-2 \leq -(z_1 + z_2) \leq 2,$$

which gives

$$\frac{-1}{2} \leq -\frac{1}{z_1 + z_2} \leq \frac{1}{2}.$$

Therefore,

$$\frac{-1}{2} \leq a_2 \leq \frac{1}{2}.$$

Hence, $|a_2| \leq \frac{1}{2}$

Converse

Now, suppose $|a_2| \leq \frac{1}{2}$. We want to prove that $f(z_1) = f(z_2)$ implies $z_1 = z_2$ for $z_1, z_2 \in \mathbb{D}$.
Assume $f(z_1) = f(z_2)$, which implies

$$z_1 + a_2 z_1^2 = z_2 + a_2 z_2^2.$$

Rearranging,

$$(z_1 - z_2) + a_2(z_1^2 - z_2^2) = 0 \implies (z_1 - z_2)(1 + a_2(z_1 + z_2)) = 0.$$

Case 1: If $z_1 = z_2$, then the function is trivially injective.
Case 2: If $z_1 \neq z_2$, then we must have:

$$1 + a_2(z_1 + z_2) = 0 \implies z_1 + z_2 = -\frac{1}{a_2}.$$

Since $z_1, z_2 \in \mathbb{D}$, we know that:

$$|z_1 + z_2| \leq |z_1| + |z_2| < 2.$$

Thus,

$$\left| -\frac{1}{a_2} \right| = \frac{1}{|a_2|}.$$

This implies

$$\frac{1}{|a_2|} < 2 \implies |a_2| > \frac{1}{2}.$$

But this contradicts the assumption that $|a_2| \leq \frac{1}{2}$. Therefore, the case where $z_1 \neq z_2$ cannot occur.

Conclusion

Thus, if $|a_2| \leq \frac{1}{2}$, the function $f(z) = z + a_2 z^2$ is injective in $\mathbb{D}$, which means it is univalent in $\mathbb{D}$.

Before proceeding to the next section of Cauchy-Riemann equations, we first define the concept of limit and derivative in $\mathbb{C}$.

Limit

Let f be a complex valued function defined at all points z in some deleted neighborhood of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, means that for each positive number ϵ , $\exists$ a positive number δ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

It is denoted by $\lim_{z \to z_0} f(z) = w_0$

Derivative

Let f be a function whose domain of definition contains a neighborhood

$$|z - z_0| < \epsilon$$

of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists
Put $\Delta z = z - z_0$,then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(\Delta z + z_0) - f(z_0)}{\Delta z}$$

Since f is defined throughout a neighborhood of z_0, the number $f(z + z_0)$ is defined for sufficiently small $|\Delta z|$
Drop the subscript on z_0 and let $\Delta w = f(z + \Delta z) - f(z)$, then

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

.

The differentiation properties which are followed in $\mathbb{R}$ are also followed in $\mathbb{C}$

Some of these properties are:-

i) $\frac{d}{dz} c = 0$
ii) $\frac{d}{dz} z = 1$
iii) $\frac{d}{dz}[cf(z)] = cf'(z)$
iv) $\frac{d}{dz} z^n = nz^{n-1}$, $n \in \mathbb{Z}$ and $z \neq 0$
v) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
vi) $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + g(z)f'(z)$
vii) $\frac{d}{dz}[\frac{f(z)}{g(z)}] = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$ for $g(z) \neq 0$
vii) $\frac{d}{dz} f(g(z)) = g'(z)f'(g(z))$
where, c is any constant in $\mathbb{C}$ and $f(z), g(z)$ are complex valued functions.

1.4 Cauchy-Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be a complex valued function , where u and v are real and imaginary parts of $f(z)$ respectively.
If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann Equations

$$u_x = v_y, \quad u_y = -v_x$$

there. Also $f'(z)$ can be written as

$$f'(z_0) = u_x + iv_y$$

where these partial derivatives are evaluated at (x_0, y_0).

In polar coordinates, these equations can be written as

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

given that first order partial derivatives of u and v exist w.r.t r and θ at $z_0 = (r_0, \theta_0)$, then $f'(z)$ exists at z_0, and

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

Sufficient Conditions for Differentiability

The satisfaction of the Cauchy–Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. But, with certain continuity conditions, we have the following useful theorem.

Theorem

Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0).

Then $f'(z_0)$ exists, its value being

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof

Assume that conditions (a) and (b) are satisfied. Write $z = x + iy$, where $0 < |z| < \epsilon$, and define

$$w = f(z_0 + z) - f(z_0).$$

Thus,

$$w = u(x_0 + x, y_0 + y) - u(x_0, y_0) + i\left(v(x_0 + x, y_0 + y) - v(x_0, y_0)\right).$$

Assuming the first-order partial derivatives of u and v are continuous at (x_0, y_0), we can write

$$u(x_0 + x, y_0 + y) - u(x_0, y_0) = u_x(x_0, y_0)x + u_y(x_0, y_0)y + \epsilon_1 x + \epsilon_2 y,$$

and

$$v(x_0 + x, y_0 + y) - v(x_0, y_0) = v_x(x_0, y_0)x + v_y(x_0, y_0)y + \epsilon_3 x + \epsilon_4 y,$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $(x, y) \to (0, 0)$.
Substituting these expressions into w, we get

$$w = u_x(x_0, y_0)x + u_y(x_0, y_0)y + \epsilon_1 x + \epsilon_2 y + i\left(v_x(x_0, y_0)x + v_y(x_0, y_0)y + \epsilon_3 x + \epsilon_4 y\right).$$

Since the Cauchy–Riemann equations are assumed to be satisfied at (x_0, y_0), we can replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in the above expression. Then we divide by $z = x + iy$ to get

$$\frac{w}{z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + \frac{\epsilon_1 + i\epsilon_3}{z}x + \frac{\epsilon_2 + i\epsilon_4}{z}y.$$

Since $|x| \le |z|$ and $|y| \le |z|$, we have

$$\left|\frac{\epsilon_1 + i\epsilon_3}{z}x\right| \le |\epsilon_1 + i\epsilon_3| \le |\epsilon_1| + |\epsilon_3|,$$

and similarly

$$\left|\frac{\epsilon_2 + i\epsilon_4}{z}y\right| \le |\epsilon_2 + i\epsilon_4| \le |\epsilon_2| + |\epsilon_4|.$$

Thus, the last two terms tend to zero as $z \to 0$. Therefore, we have established the expression for $f'(z_0)$.

Examples

(a) Consider the exponential function

$$f(z) = e^z = e^x e^{iy} \quad (z = x + iy),$$

In view of Euler's formula, this function can be written as

$$f(z) = e^x \cos y + ie^x \sin y,$$

Therefore,

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere, and since these derivatives are continuous everywhere, the conditions in the above theorem are satisfied at all points in the complex plane. Thus, $f'(z)$ exists everywhere, and

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that

$$f'(z) = f(z) \quad \text{for all } z.$$

(b) It follows from our theorem that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

has a derivative at $z = 0$. In fact,

$$f'(0) = 0 + i0 = 0.$$

and this function cannot have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points.

1.5 Analytic functions

Definition

A function f of the complex variable z is said to be analytic at a point z_0 if it has a derivative at every point within a neighborhood around z_0. Conversely, if f is not analytic at z_0, yet there is a point in every neighborhood of z_0 where f is analytic, then z_0 is termed a singular point or point of singularity for f. Consider the function $f(z) = 1/z$; it is analytic at every nonzero point in the finite plane. However, the function $f(z) = |z|^2$ is not analytic at any point because it only possesses a derivative at $z = 0$ and lacks a derivative throughout any neighborhood. When a function is analytic at every point across the entire finite plane, it is classified as entire. Entire functions include polynomial functions, exponential functions, and the sine and cosine functions. Additionally, one can also use the properties of derivatives to demonstrate that a function is analytic.

Examples

(a) $f(z) = \frac{z^3+i}{z^2-3z+2}$

To determine the singular points of this function, we find those values of z which are the solution of $z^2 - 3z + 2 = 0$. These values are $z = 1$ and $z = 2$, from which it follows that $f(z)$ is differentiable everywhere except at $z = 1, 2$.

Therefore $z = 1, 2$ are singular points of $f(z) = \frac{z^3+i}{z^2-3z+2}$.

(b) $h(z) = 3x + y + i(3y - x)$ is entire

Here, $u = 3x + y \quad$ and $\quad v = 3y - x$

Both u and v are polynomials, hence their first order partial derivatives exist, w.r.t. x and y .

Also

$$u_x = 3 = v_y \ , \ u_y = 1 = -v_x$$

This proves that $h(z)$ is entire.

(c) $F(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$ is not entire

Let $q(z) = (z + 2)(z^2 + 2z + 2)$

Note that,

$$z^2 + 2z + 2 = (z + 1)^2 + 1$$

Values of z satsifying $q(z) = 0$, are

$$z = -2; -1 \pm i$$

$\Rightarrow F(z)$ is not analytic throughout its domain, and hence not entire.

(d) $G(z) = e^{-y}\sin x - ie^{-y}\cos x, \quad \text{is entire}$

$$u = e^{-y}\sin x \quad v = -e^{-y}\cos x$$

$e^{-y}, \sin x$ and $\cos x$ are differentiable functions, hence $e^{-y}\sin x$ and $-e^{-y}\cos x$ are also differentiable.

$\Rightarrow$ First order partial derivatives of u and v exist w.r.t x and y.

and

$$u_x = e^{-y}\cos x \quad u_y = -e^{-y}\sin x$$
$$v_x = e^{-y}\sin x \quad v_y = e^{-y}\cos x$$

$$\Rightarrow u_x = v_y \quad u_y = -v_x$$

By sufficient condition of differentiability, we infer $G(z)$ is entire.

1.6 Starlike Functions

A function $w = f(z)$ is called a **starlike function** if it satisfies the following conditions:

- The function $f(z)$ is regular (analytic) and univalent (injective) in the unit disk, i.e., for $|z| < 1$.
- The image of the disk $|z| < 1$ under the function $f(z)$ is a **starlike domain** with respect to w_0. This means that for any point w_0 in the image, the line segment joining w_0 to any other point in S lies entirely within S.

The function f is said to be *starlike* if it maps $\mathbb{U} = \{z : |z| < 1\}$ onto a domain that is star-shaped with respect to the origin. This is equivalent to the condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for} \quad |z| < 1.$$

This condition guarantees that the image of the disk under f is a starlike domain.

Starlike Class

The set of all **starlike functions** is denoted by $\mathcal{S}^*$. A starlike function is a function that preserves the starlike nature of a domain with respect to the origin. Specifically, we define:

$$\mathcal{A} = \{f : D \to \mathbb{C} \mid f \text{ is analytic and univalent, } f(0) = 1,\ f'(0) = 1\}$$

The set $\mathcal{S}^*$ consists of all functions in $\mathcal{A}$ that satisfy the following condition for starlikeness:

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} \mid \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ for all } z \in \mathbb{U} \right\}.$$

Here, $\Re(\cdot)$ denotes the real part, and the condition $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ ensures that the image of the function f remains starlike with respect to the origin for all z in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.

Examples

The following functions are starlike as evident form the line segment formed.

(a) $f(z) = \frac{z}{2} + \frac{z^{13}}{37}$

This function is starlike with respect to $f(0) = 0$

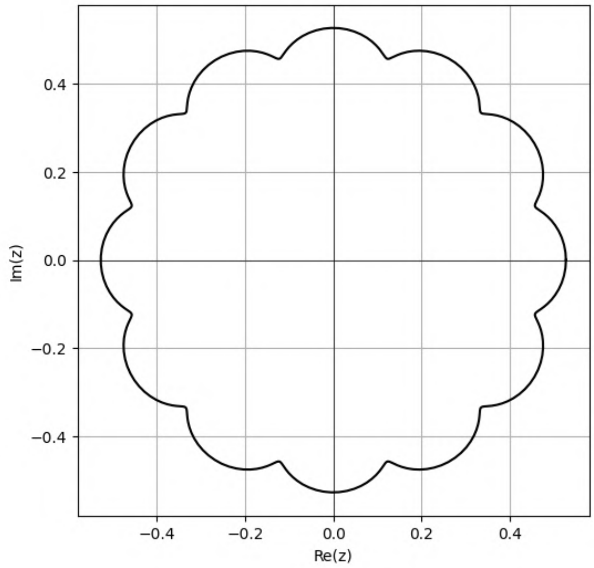


Figure 8:

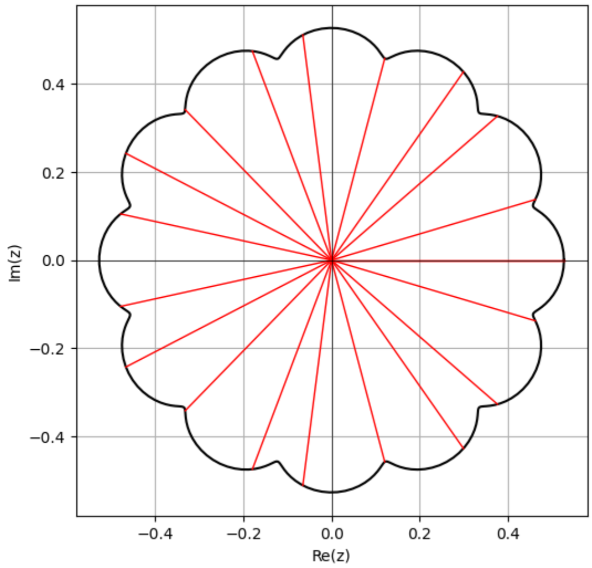


Figure 9:

(b) $g(z) = 1 + \tan z$

This function is starlike with respect to $g(0) = 1$

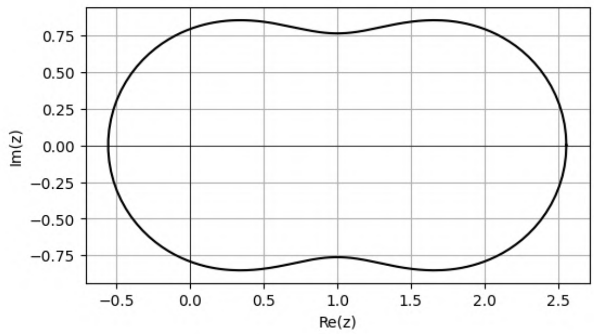


Figure 10:

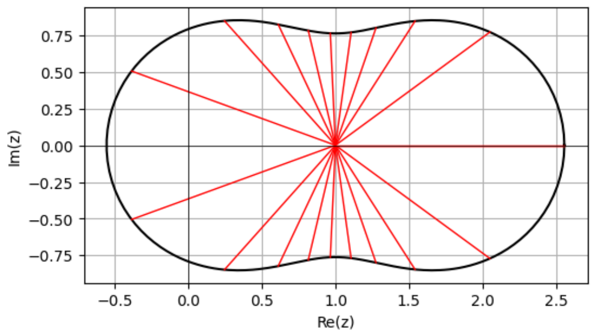
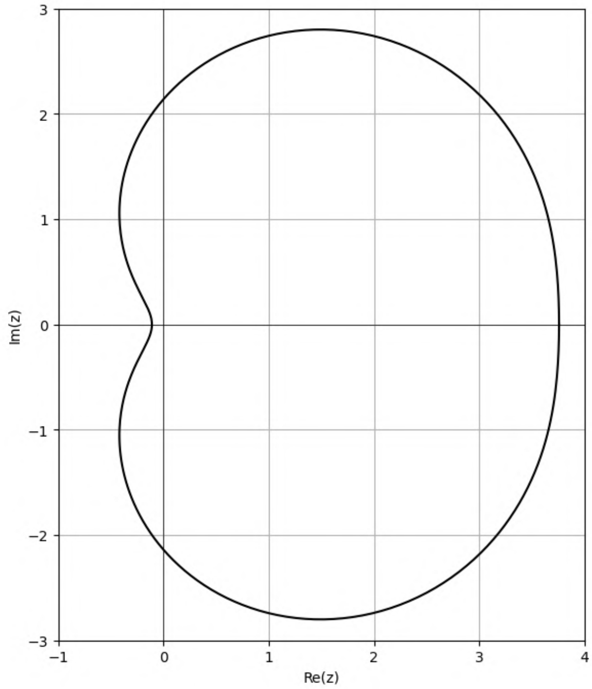


Figure 11:

(c) $h(z) = e^z(\sin z + \cos z)$

This function is starlike with respect to $h(0) = 1$



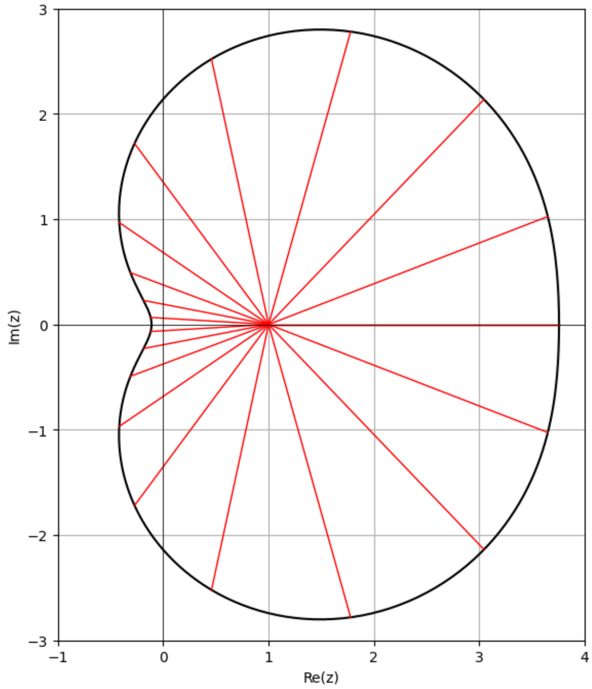
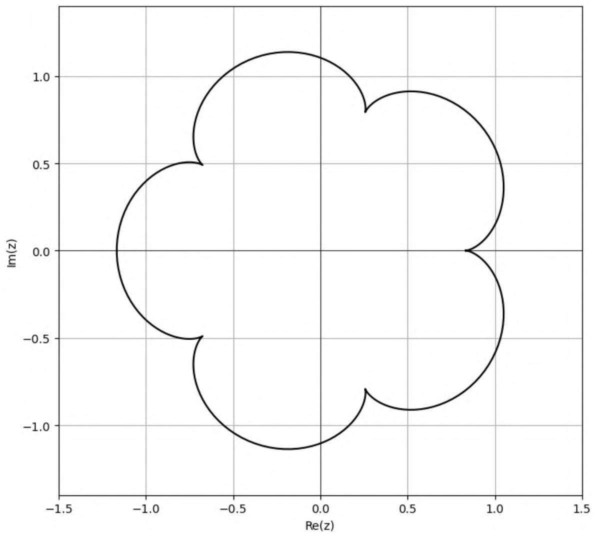


Figure 12: Figure 13:

(d) $\theta(z) = z - \frac{z^6}{6}$

This function is starlike with respect to $\theta(0) = 0$



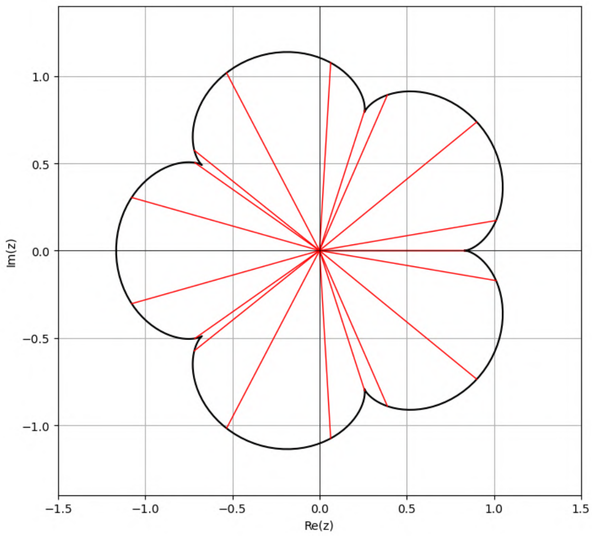
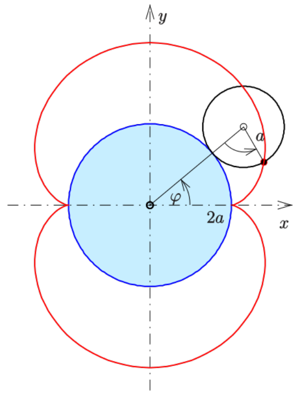


Figure 14: Figure 15:

Chapter 2

Function associated with Nephroid domain

2.1 Introduction



The name nephroid (meaning 'kidney shaped') was used for the two-cusped epicycloid by Proctor in 1878. The nephroid is the epicycloid formed by a circle of radius a rolling externally on a fixed circle of radius $2a$. The nephroid has length $24a$ and area $12\pi a^2$ and is given by the parametric equations:

$$x = a(3\cos t - \cos 3t)$$
$$y = a(3\sin t - \sin 3t)$$

The evolute of a nephroid is another nephroid half as large and rotated 90 degrees. This figure is about nephroid and its evolute magenta: point with osculating circle and center of curvature . As the evolute of a nephroid is another nephroid, the involute of the nephroid is also another nephroid. The original nephroid in the image is the involute of the smaller nephroid.

The function considered in this chapter is

$$\eta(z) = 1 + \frac{z}{2} + \frac{z^3}{6} \quad \text{, where } z \in \mathbb{D}$$

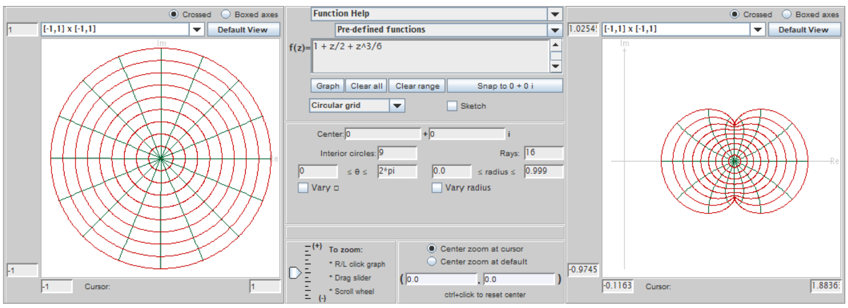


Figure 1: Image of unit disk under the function $\eta(z) = 1 + \frac{z}{2} + \frac{z^3}{6}$

The function $\eta(z)$ satisfies the following:-

1.$\eta(z)$ is univalent
2.$Re(\eta) > 0$
3.$\eta(\mathbb{D})$ is starlike with respect to $\eta(0) = 1$
4.$\eta(\mathbb{D})$ is symmetric about the real axis
5.$\eta'(0) > 0$

We now prove that η satifies each of these properties

1. Let $z_1, z_2 \in \mathbb{D}$ and let $\eta(z_1) = \eta(z_2)$,then

$$\begin{aligned}
1 + \frac{z_1}{2} + \frac{z_1^3}{2} &= 1 + \frac{z_2^2}{2} + \frac{z_2^2}{2} \\
\Rightarrow \frac{z_1}{2} - \frac{z_2}{2} + \frac{z_1^3}{6} - \frac{z_2^3}{6} &= 0 \\
\Rightarrow \frac{z_1 - z_2}{2} + \frac{z_1^3 - z_2^3}{6} &= 0 \\
\Rightarrow 3(z_1 - z_2) + (z_1^3 - z_2^3) &= 0 \\
\Rightarrow (z_1 - z_2)(3 + z_1^2 + z_2^2 + z_1 z_2) &= 0
\end{aligned}$$

Note that since $z \in \mathbb{D}$, hence

$$(3 + z_1^2 + z_2^2 + z_1 z_2) \neq 0$$

So, $z_1 - z_2 = 0$ is true, which implies, $z_1 = z_2$

Therefore $\eta(z_1) = \eta(z_2) \Rightarrow z_1 = z_2$, hence η is univalent in $\mathbb{D}$

2. Put $z = re^{i\theta}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ then

$$\eta(re^{i\theta}) = 1 + \frac{re^{i\theta}}{2} + \frac{r^3 e^{3i\theta}}{6}$$

$$\Rightarrow \eta(re^{i\theta}) = 1 + \frac{r\cos\theta}{2} + \frac{r^3\cos 3\theta}{6} + i\left(\frac{r\sin\theta}{2} + \frac{r\sin 3\theta}{6}\right)$$

$$\Rightarrow Re(\eta) = 1 + \frac{r\cos\theta}{2} + \frac{r\cos 3\theta}{6}$$

Note that, for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$

$$\begin{aligned}
-1 &\leq \cos\theta \leq 1 \\
\Rightarrow -r &\leq r\cos\theta \leq r \\
\Rightarrow 0 &\leq r\cos\theta \leq 1 \\
\Rightarrow 0 &\leq \frac{r\cos\theta}{2} \leq \frac{1}{2}
\end{aligned}$$

Similarly, we can show that $0 \leq \frac{r^3\cos 3\theta}{6} \leq \frac{1}{6}$

On adding 1 along with both the inequalities, we get

$$1 \leq 1 + \frac{r\cos\theta}{2} + \frac{r^3\cos 3\theta}{6} \leq \frac{5}{6}$$

i.e

$$1 \leq Re(\eta) \leq \frac{5}{3}$$

Therefore, $Re(\eta) > 0$

3. To prove $\eta(\mathbb{D})$ is starlike with respect to $\eta(0) = 1$, we will show that

$$Re\left(\frac{z\eta'(z)}{\eta(z) - 1}\right) > 0$$

To compute this, we use mathematica software, by which we get,

$$Re\left(\frac{z\eta'(z)}{\eta(z) - 1}\right) = \frac{3(3 + r^4 + 4r^2\cos 2\theta)}{9 + r^4 + 6r^2\cos 2\theta}$$

where we have taken $z = re^{i\theta}$

Since, $z \in \mathbb{D}$, it is evident that above equation is greater than 0

Also, on plotting $Re\left(\frac{z\eta'(z)}{\eta(z)-1}\right)$ in mathematica, we get

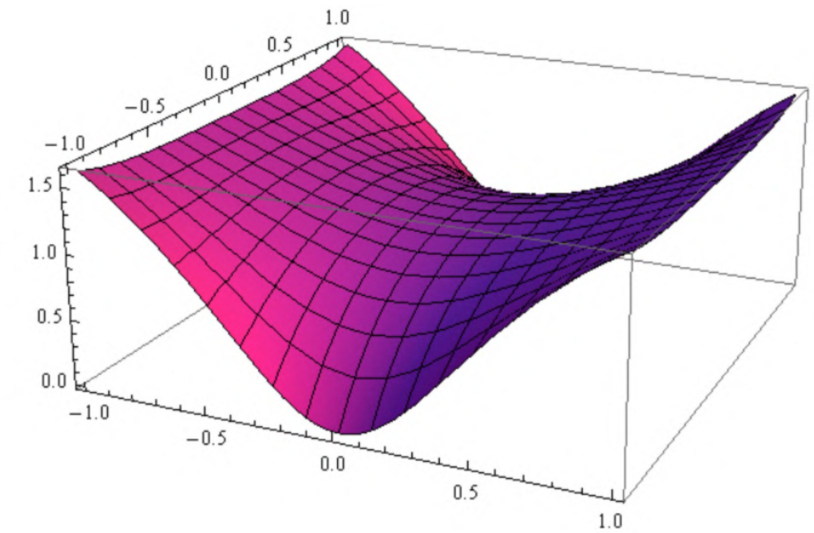


Figure 2: Real part of complex function $\frac{z\eta'(z)}{\eta(z)-1}$

Which shows that $Re\left(\frac{z\eta'(z)}{\eta(z)-1}\right) > 0$

4. The graph of $\eta(z)$ shows that the image formed above and below real axis are same, therefore, we can say that $\eta(z)$ is symmetric about real axis over $\mathbb{D}$, i.e, $\eta(\mathbb{D})$ is symmetric about real axis

5. $\eta'(z) = \frac{1}{2} + \frac{z^2}{2}$ and as $\eta'(0) = \frac{1}{2}$, hence

$$\eta'(0) > 0$$

2.2 Main Results

Theorem 2.1. The function $\eta(z) = 1 + \frac{z}{2} + \frac{z^3}{6}$ maps $\mathbb{D}$ onto the region bounded by

$$\left(3\left(\left((u-1)^2\right) + v^2 - \frac{1}{9}\right)\right)^3 - \left(\frac{3}{2}(u-1)\right)^2 = 0$$

Proof. Let $t \in (-\pi, \pi]$ and put $z = e^{it}$, we get

$$u + iv = \eta(e^{it}) = 1 + \frac{e^{it}}{2} + \frac{e^{3it}}{6}$$

On expanding e^{it} and e^{3it},

$$u + iv = 1 + \frac{\cos t + i \sin t}{2} + \frac{\cos 3t + i \sin 3t}{6}$$

Therefore,

$$u = 1 + \frac{\cos t}{2} + \frac{\cos 3t}{6}, v = \frac{\sin t}{2} + \frac{\sin 3t}{6}$$

Note that,

$$\begin{aligned}(u-1)^2 + v^2 &= \frac{1}{4} + \frac{1}{36} + \frac{\cos 2t}{6} \\ &= \frac{1}{4} + \frac{1}{36} - \frac{1}{6} + \frac{\cos^2 t}{3} \\ \Rightarrow 3\left((u-1)^2 + v^2 - \frac{1}{9}\right) &= \cos^2 t\end{aligned}$$

and

$$\begin{aligned}u &= 1 + \frac{2\cos^3 t}{6} \\ \Rightarrow \cos t &= \left(\frac{3(u-1)}{2}\right)^{\frac{1}{3}}\end{aligned}$$

Therefore,

$$3\left((u-1)^2 + v^2 - \frac{1}{9}\right) = \left(\frac{3(u-1)}{2}\right)^{\frac{2}{3}}$$

$$\Rightarrow \left(3\left((u-1)^2 + v^2 - \frac{1}{9}\right)\right)^3 - \left(\frac{3(u-1)}{2}\right)^2 = 0$$

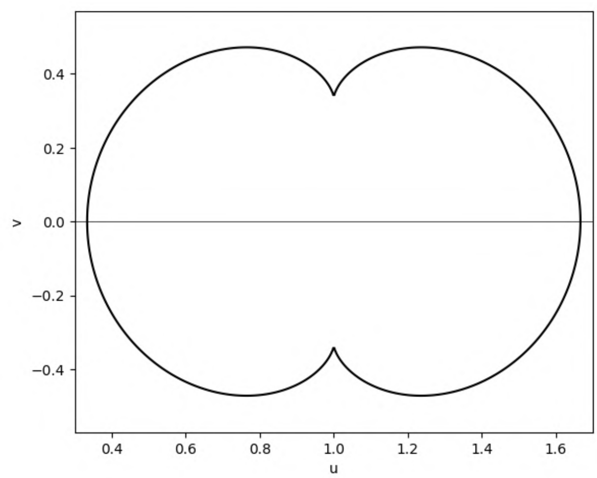


Figure 3: Boundary of $\eta(z)$

Theorem 2.2 For $0 < r < 1$, the function $\eta(z)$ satisfies

$$\min_{|z|=r} Re(\eta(z)) = 1 - \frac{r}{2} - \frac{r^3}{6}$$

and,

$$\max_{|z|=r} Re(\eta(z)) = 1 + \frac{r}{2} + \frac{r^3}{6}$$

<u>**Proof**</u>. Let $z = re^{it}$, where $0 < r < 1$, and $-\pi < t \leq \pi$,then

$$Re(\eta(re^{it})) = 1 + \frac{r\cos t}{2} + \frac{r^3 \cos 3t}{6}$$

$$= 1 + \frac{r\cos t}{2} + \frac{r^3(4\cos^3 t - 3\cos t)}{6}$$

$$= 1 + \frac{r\cos t(1 - r^2)}{2} + \frac{2r^3\cos^3 t}{3}$$

$$= 1 + \frac{rx(1 - r^2)}{2} + \frac{2r^3x^3}{3} = f(x) \quad , x = cost$$

On taking derivative of f, we get

$$f'(x) = \frac{1}{2}r(1 - r^2) + 2r^3x^3$$

The values satisfying $f'(x) = 0$ are

$$x = \frac{1}{2r}\sqrt{r^2 - 1} \quad \text{and} \quad x = -\frac{1}{2r}\sqrt{r^2 - 1}$$

which are both complex as $0 < r < 1$, therefore, we find the values of f at its boundary points, i.e, at $x = -1$ and $x = 1$

$$f(-1) = 1 - \frac{r}{2} - \frac{r^3}{6}$$

$$f(1) = 1 + \frac{r}{2} + \frac{r^3}{6}$$

On plotting these values, we get

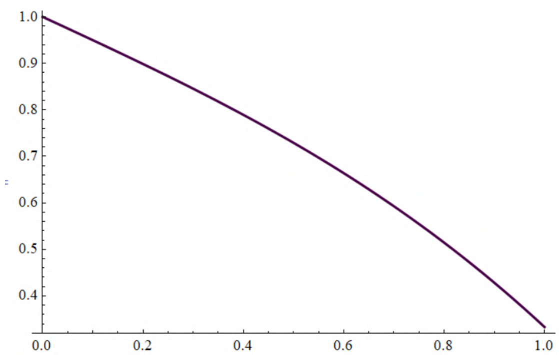


Figure 4: $f(-1)$

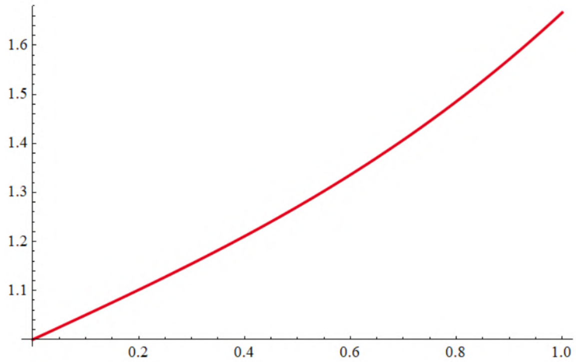


Figure 5: $f(1)$

From the above graphs, we infere that $f(1)$ is an increasing function of r, whereas $f(-1)$ is a decreasing function of r, $\forall\ 0 < r < 1$

Therefore,

$$\min_{|z|=r} Re(\eta(z)) = f(-1) = 1 - \frac{r}{2} - \frac{r^3}{6}$$

$$\max_{|z|=r} Re(\eta(z)) = f(1) = 1 + \frac{r}{2} + \frac{r^3}{6}$$

Theorem 2.3 Let $\frac{1}{3} < a < \frac{5}{3}$. Let r_a and R_a be given by,

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a < \frac{4}{5} \\ \sqrt{(a-1)^2 + \frac{1}{9}} & \text{if } \frac{4}{5} < a < \frac{6}{5} \\ \frac{5}{3} - a & \text{if } \frac{6}{5} < a < \frac{5}{3} \end{cases}$$

$$R_a = \begin{cases} \frac{5}{3} - a & \text{if } \frac{1}{3} < a < 1 \\ a - \frac{1}{3} & \text{if } 1 < a < \frac{5}{3} \end{cases}$$

then

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subseteq \Gamma \subseteq \{w \in \mathbb{C} : |w - a| < R_a\}$$

where $\Gamma = \{u + iv : \left(3\left((u-1)^2 + v^2 - \frac{1}{9}\right)\right)^3 - \left(\frac{3(u-1)}{2}\right)^2 < 0\}$

Proof. Let $z = e^{it}$,then

$$u(t) = 1 + \frac{\cos t}{2} + \frac{cos3t}{6}\ ,\ v(t) = \frac{\sin t}{2} + \frac{\sin 3t}{6},\ \ -\pi < t \leq \pi$$

where u and v are real and imaginary parts of η respectively

The square of the distance from the point $(a, 0)$ to the points on the boundary equation of η is given by,

$$\psi(t) = (a - u(t))^2 + v(t)^2 = (a-1)^2 + \frac{1}{9} + \frac{cos^2 t}{3} - \frac{4(a-1)cos^3 t}{3}$$

$$= (a-1)^2 + \frac{1}{9} + \frac{x^2}{3} - \frac{4(a-1)x^3}{3}$$
$$= g(x)$$

$x = \cos t, -\pi < t \leq \pi$

But since the boundary equation of η is symmetric about real axis(by property 4), we can consider only $0 \leq t \leq \pi$

Now, $g'(x) = 0 \Rightarrow x = 0$ or $x = \frac{1}{6(a-1)}$

But since $\lim_{a \to 1} \frac{1}{6(a-1)} = \infty$

Hence, $\frac{1}{6(a-1)}$ cannot be a crtical point for g

At $x = 0$, $g''(0) = \frac{2}{3}$

$\Rightarrow x_0 = 0$ is point of absolute minima for g

Now we compare the values of g at its end points to find local maximum and local minimum

At $x = -1$

$$g(-1) = \left(a - \frac{1}{3}\right)^2$$

At $x = 1$

$$g(1) = \left(\frac{5}{3} - 1\right)^2$$

Also as g attains absolute minima at x_0, so

$$g(0) = (a-1)^2 + \frac{1}{9}$$

Note that, for $\frac{1}{3} < a < \frac{4}{5}$

$$g(-1) < g(0), g(1)$$

for $\frac{4}{5} < a < \frac{6}{5}$

$$g(0) \leq g(-1), g(1)$$

and for $\frac{6}{5} < a < \frac{5}{3}$

$$g(1) < g(0), g(-1)$$

Also, for $\frac{1}{3} < a < 1$,

$$g(-1) < g(1)$$

and

$$g(-1) < g(1), \text{ for } 1 < a < \frac{5}{3}$$

$$\min_{0\le t\le \pi} \psi(t) = min\{g(-1), g(0), g(1)\}$$

and,

$$\max_{0\le t\le \pi} \psi(t) = max\{g(-1), g(1)\}$$

Therefore, we have

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a < \frac{4}{5} \\ \sqrt{(a-1)^2 + \frac{1}{9}} & \text{if } \frac{4}{5} < a < \frac{6}{5} \\ \frac{5}{3} - a & \text{if } \frac{6}{5} < a < \frac{5}{3} \end{cases}$$

$$R_a = \begin{cases} \frac{5}{3} - a & \text{if } \frac{1}{3} < a < 1 \\ a - \frac{1}{3} & \text{if } 1 < a < \frac{5}{3} \end{cases}$$

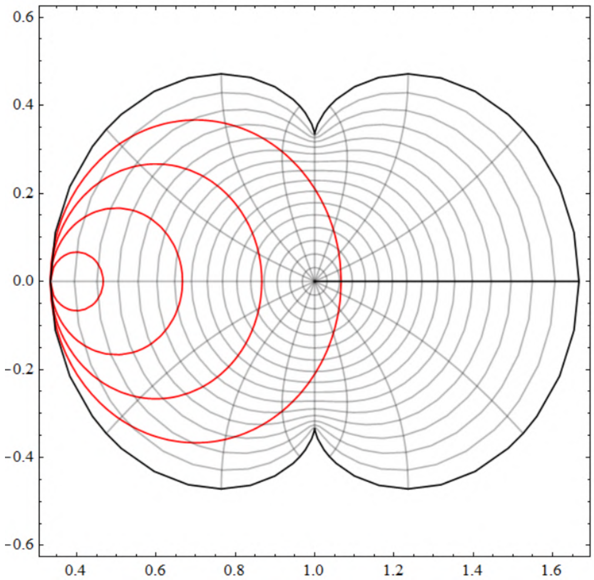


Figure 6: $r_a = a - \frac{1}{3}$

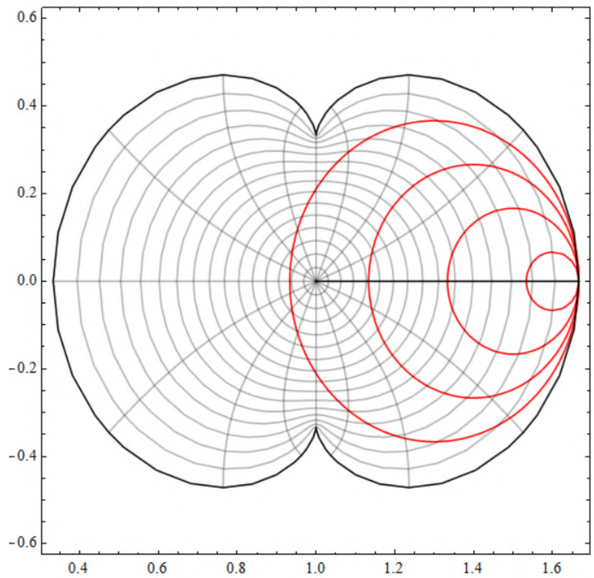


Figure 7: $r_a = \frac{5}{3} - a$

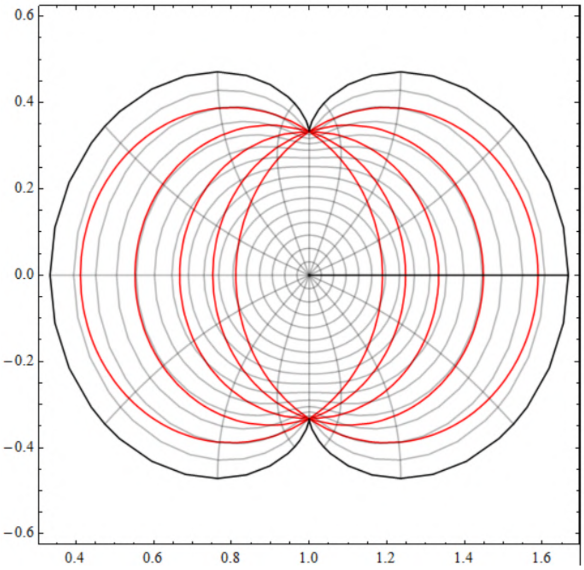


Figure 8: $r_a = \sqrt{(a-1)^2 + \frac{1}{9}}$

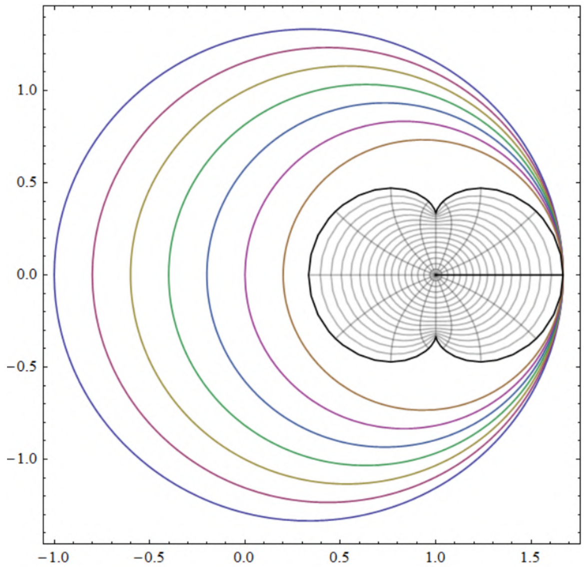


Figure 9: $R_a = \frac{5}{3} - a$

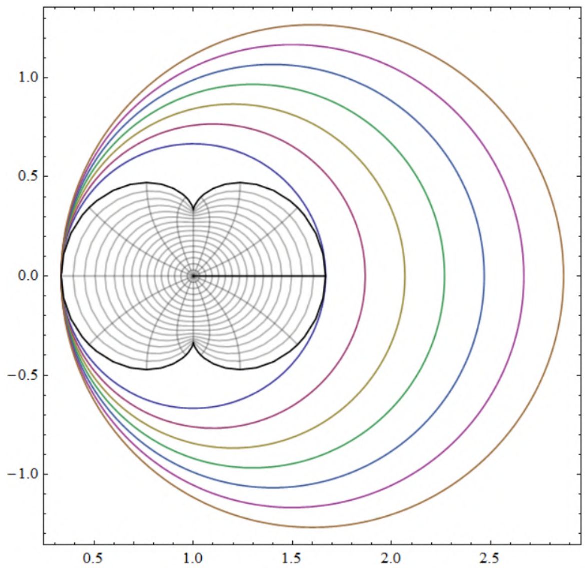


Figure 10: $R_a = a - \frac{1}{3}$

We now use the following theorem's statement to construct starlike functions using analytic functions

Theorem 2.4 A function belongs to class $S^*(\eta)$ iff $\exists$ an analytic function $p(z)$, satisfying $p(z) \prec \eta(z)$, such that

$$f(z) = z \exp\left(\int_0^z \frac{p(\xi)-1}{\xi} d\xi\right), \quad z \in \mathbb{D}$$

where $S^*(\eta) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \eta(z) \right\}$ is a subclass of starlike functions .

Consider the following analytic functions $p_i : \mathbb{D} \to \mathbb{C}$, $i = 1, 2, 3, 4, 5$, defined as follows.

i) $p_1(z) = 1 + \frac{z}{3} + \frac{z^7}{38}$

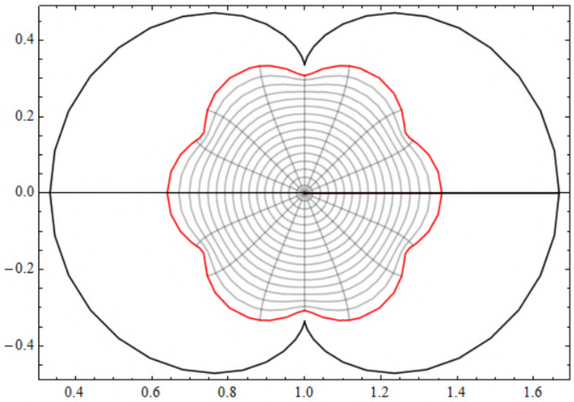


Figure 11: Image of $p_1(z)$ inside $\eta(z)$

ii) $p_2(z) = 1 + \frac{ze^{\frac{z}{2}}}{4}$

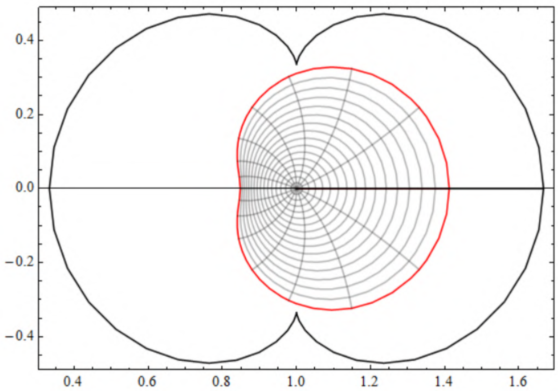


Figure 12: Image of $p_2(z)$ inside $\eta(z)$

iii)$p_3(z) = 1 + \frac{z\cosh(z)}{10} + \frac{z\sinh(z)}{15} + \frac{z}{4}$

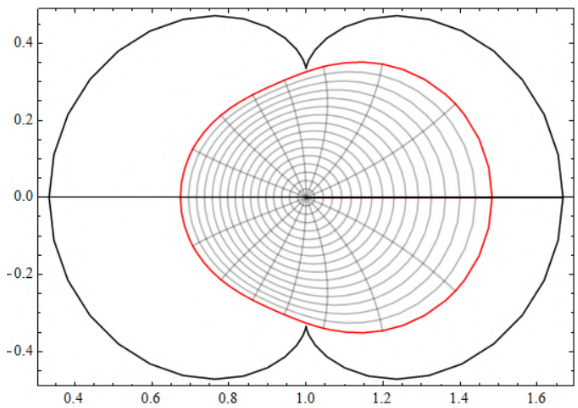


Figure 13: Image of $p_3(z)$ inside $\eta(z)$

iv)$p_4(z) = 1 + \frac{z}{4} + \frac{z^5}{19}$

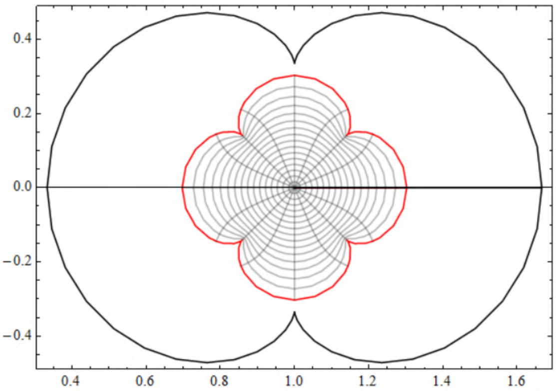


Figure 14: Image of $p_4(z)$ inside $\eta(z)$

v) $p_5(z) = 1 + \frac{z}{4} + \frac{z^2}{10} + \frac{z^5}{25} + \frac{z^6}{40}$

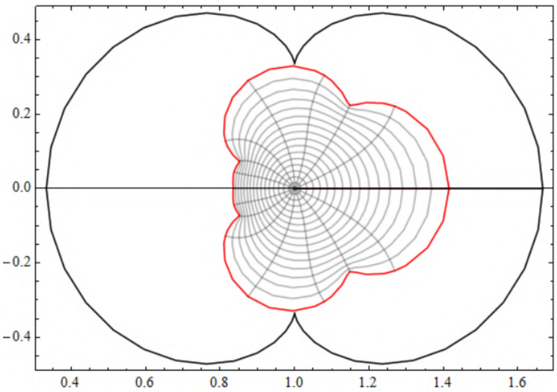


Figure 15: Image of $p_5(z)$ inside $\eta(z)$

For each $i = 1, 2, 3, 4, 5$, $p_i(0) = 1$ and $p_i(\mathbb{D}) \subseteq \eta(\mathbb{D})$ and $\eta(z)$ is univalent, hence $p(z) \prec \eta(z)$

By using theorem, the following functions are members of S^*

i) $f_1(z) = z \exp\left(\frac{z}{3} + \frac{z^7}{266}\right)$

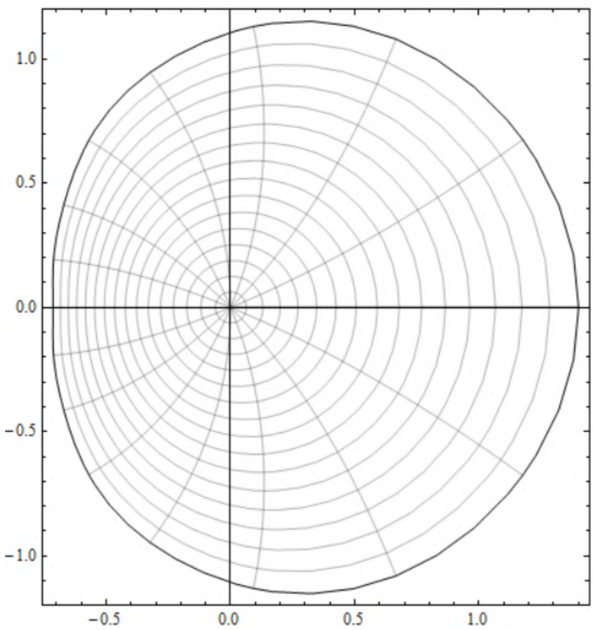


Figure 16: $f_1(z)$

ii) $f_2(z) = z\ \exp\left(\frac{1}{2}(-1 + e^{\frac{z}{2}})\right)$

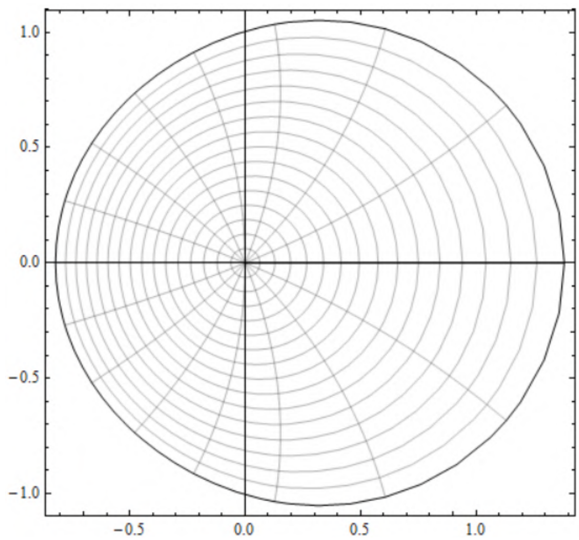


Figure 17: $f_2(z)$

iii) $f_3(z) = z \exp\left(\frac{1}{60}(6\sinh(z) + 4\cosh(z) + 15z - 4)\right)$

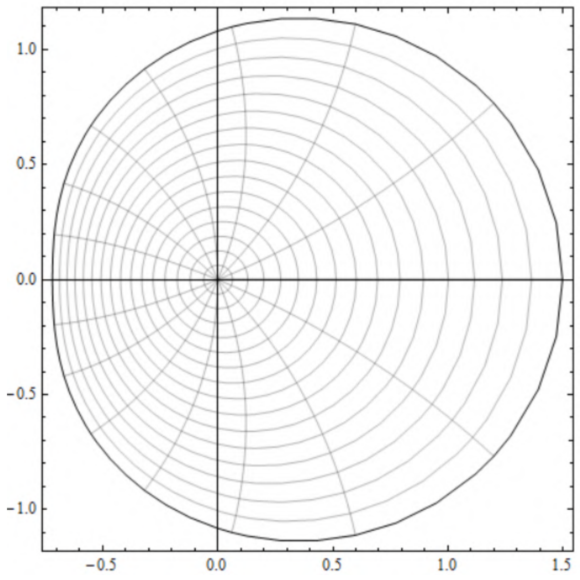


Figure 18: $f_3(z)$

iv) $f_4(z) = z\exp\left(\frac{z}{4} + \frac{z^5}{95}\right)$

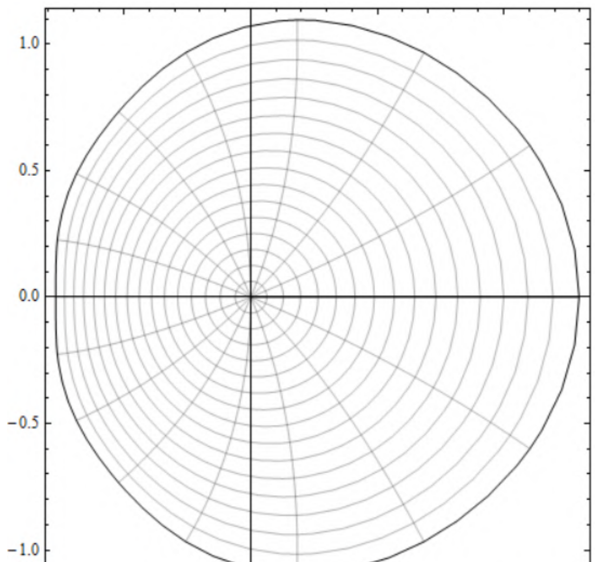


Figure 19: $f_4(z)$

v) $f_5(z) = z\exp\left(\frac{z}{4} + \frac{z^2}{20} + \frac{z^5}{125} + \frac{z^6}{240}\right)$

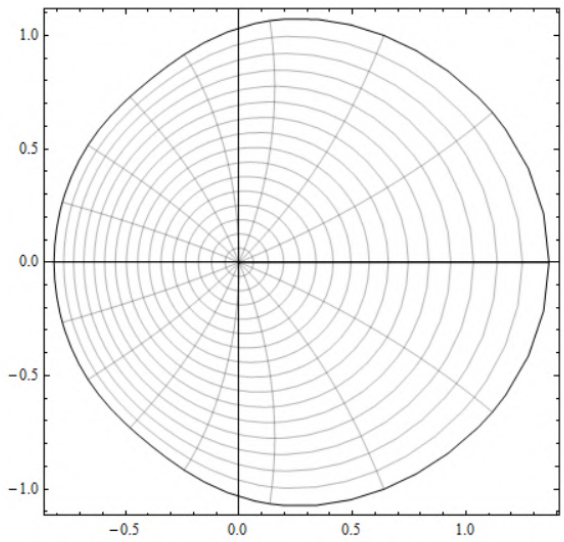


Figure 20: $f_5(z)$

The plot of each $f_i(z)$ (for $i = 1, 2, 3, 4, 5$), verifies that they are starlike.

If we choose $p(z) = \eta(z) = 1 + \frac{z}{2} + \frac{z^3}{6}$, then

$$f(z) = ze^{\frac{z}{2}+\frac{z^3}{18}} = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{11z^4}{144} + \frac{35z^5}{1152} + \frac{83z^6}{11520} + \frac{1129z^7}{414720} + ...$$

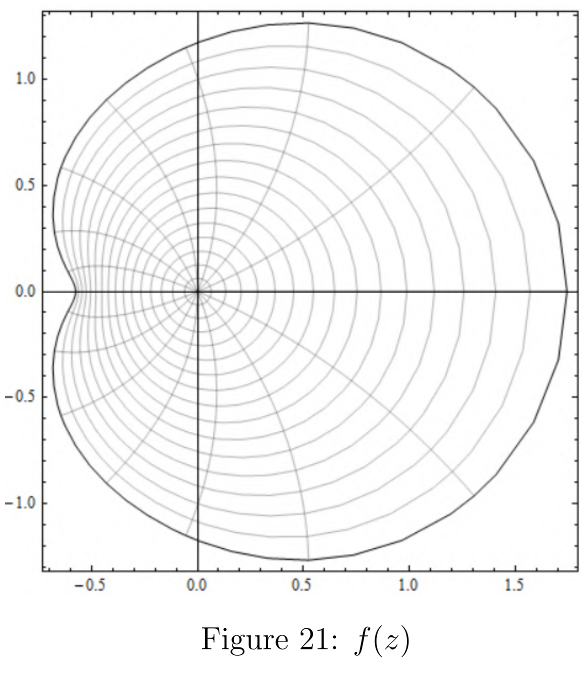


Figure 21: $f(z)$

2.3 Conclusion

In this chapter, several key results have been derived for the function $\eta(z) = 1 + \frac{z}{2} + \frac{z^3}{6}$. The function satisfies $\eta(0) = 1$, and its real part is always postive. Additionally, the image of the unit disk under $\eta'(z)$ is starlike with respect to 1 and is symmetric about the real axis. The derivative at the origin, $\eta'(0)$ is postitive.

The boundary of the function's image is described by the equation

$$\left(3\left(\left((x-1)^2\right)+y^2-\frac{1}{9}\right)\right)^3-\left(\frac{3}{2}(x-1)\right)^2=0$$

Furthermore, the minimum and maximum values of $Re(\eta(z))$ is determined by

$$1-\frac{r}{2}-\frac{r^3}{6} \text{ and } 1+\frac{r}{2}+\frac{r^3}{6}$$

respectively for $0<r<1$ Finally, the smallest disk centered at $z=1$ that contains Γ is $\{w \in \mathbb{C} : |w-1| < \frac{2}{3}\}$ and the largest disk centered at $z=1$ that is contained in Γ is $\{w \in \mathbb{C} : |w-1| < \frac{1}{3}\}$

Chapter 3

Function associated with a sum of exponential and polynomial function

3.1 Introduction

A limacon is a plane curve that can be defined as the path of a point fixed to a circle that rolls around another circle of the same radius. The word 'limacon' comes from French and literally means 'snail'. They belong to the family of curves called centered trochoids; more specifically, they are epitrochoids. The polar equation for a limaçon is r = a ± b $\sin t$ or r = a ± b $\cos t$, where a and b are not equal to zero.

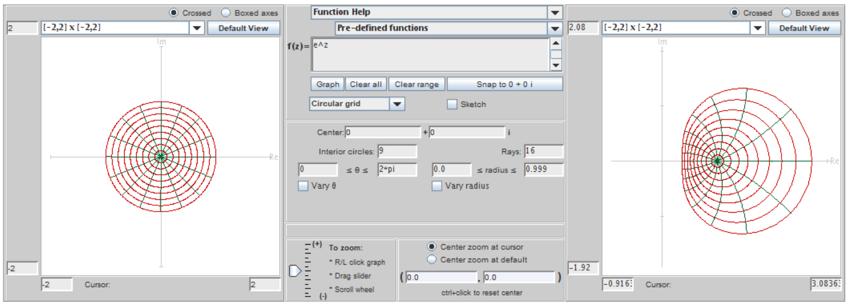


Figure 1: Image of D under $f(z) = e^z$

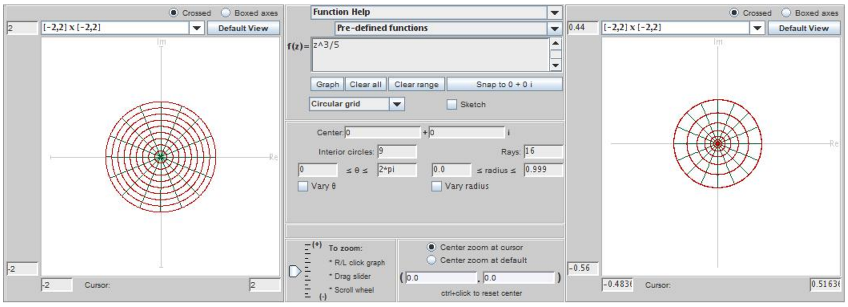


Figure 2: Image of D under $f(z) = \frac{z^3}{5}$

A complex function $f(z) = e^z$ is associated with Convex Limacon domain. A convex limacon is a plane curve that looks like a circle that has been flattened on one side. However, a complex function $f(z) = \frac{z^3}{5}$ is associated with Circle domain.

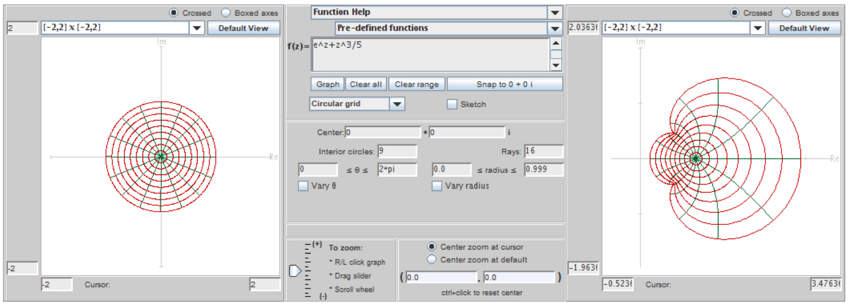


Figure 3: Image of D under $f(z) = e^z + \frac{z^3}{5}$

- f is univalent in $\mathbb{D}$ as the image curve does not have any intersecting lines. Also, $Re(f) > 0$.
- $f(0) = 1$ and f is starlike w.r.t. $f(0) = 1$.
- f is symmetric about the real axis.
- $f'(z) = e^z + \frac{3z^2}{5}$
  Hence, $f'(0) = 1 > 0$

We note that the function $f(z) = e^z + \frac{z^3}{5}$ maps $\mathbb{D}$ onto the region which is symmetric about the real axis and lies completely in the right-half plane.

3.2 Main Results

Theorem 1. *For $\sqrt{2} - 1 < a \leq \sqrt{2} + 1$, let $r_a = 1 - |\sqrt{2} - a|$ and $R_a = \sqrt{a^2 + 1}$. Then*

$$\{w : |w - a| < r_a\} \subset \{w : |w - a| < R_a\}.$$

Proof. For $z = e^{it}$, the parametric equations of $f(z) = e^z + \frac{z^3}{5}$ are

$$u(t) = e^{cost} cossint + \frac{cos3t}{5}$$

and

$$v(t) = e^{cost} sinsint + \frac{sin3t}{5},$$

$-\pi < t < \pi$.
The square of the distance from the point (a,0) to the points on boundary of $f(\mathbb{D})$ is given by

$$z(t) = (a - u(t))^2 + (v(t))^2$$

$$= (a - (e^{cost} cossint + \frac{cos3t}{5}))^2 + (e^{cost} sinsint + \frac{sin3t}{5})^2$$

$$= a^2 + \frac{1}{25} + e^{2cost} + \frac{2}{5} e^{cost} cos(3t - sint) - 2ae^{cost} cossint - \frac{2}{5} acos3t$$

It can be easily seen that

$$z'(t) = -2e^{2cost}sint - \frac{6}{5}e^{cost}sint(3t - sint) + \frac{6}{5}asin3t - \frac{2}{5}e^{cost}sin(sint - 2t) + 2ae^{cost}sin(t + sint)$$

A calculation shows that $z'(t) \neq 0$ for any $0 < t < \pi$.

Plotting the $z'(t)$ over t from 0 to 2π gives the critical points graphically. The points on x-axis where the graph cuts axis are the possible critical points.
Critical points when $z'(t) = 0$ are: $\{0, 0.756252, 2.02816, 3.14159 = \pi, 4.25502, 5.52693\}$.

Then, we can check the nature of critical points and easily find whether they are a maximum or a minimum.

Finally, we will plot the Minima, Maxima and Radius of function $f(z) = e^z + \frac{z^3}{5}$ using *Desmos Online Graphing Calculator*.

$\square$

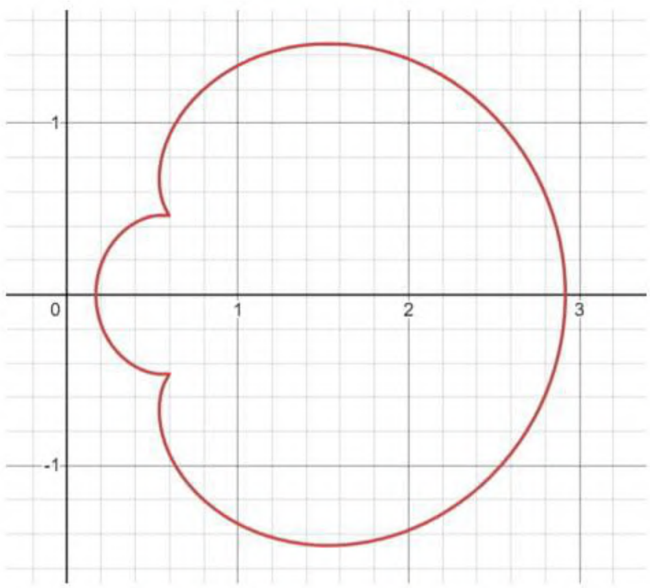


Figure 4: Image of disc under $f(z) = e^z + \frac{z^3}{5}$

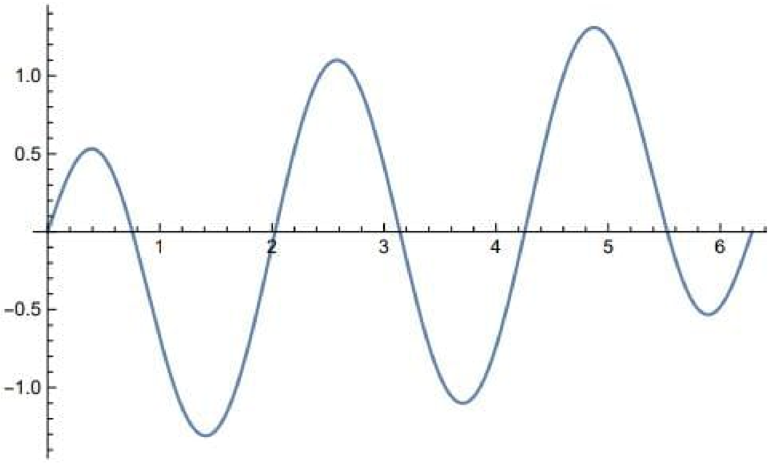


Figure 5: Plotting of z'(t)

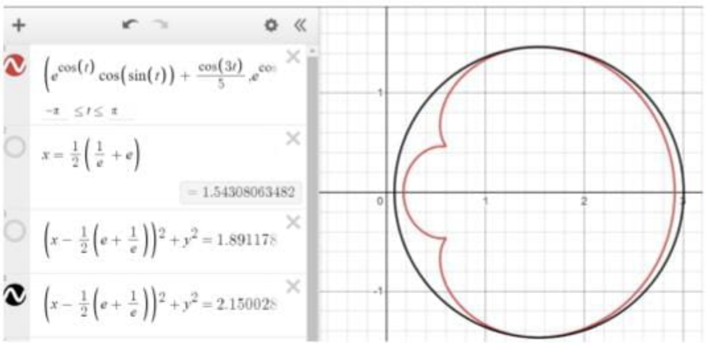


Figure 6: Maxima of $f(z) = e^z + \frac{z^3}{5}$

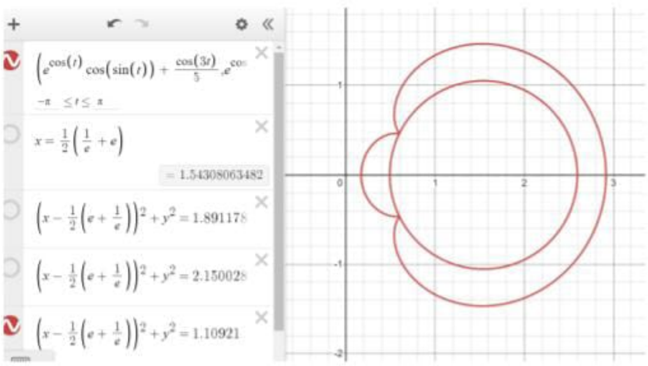


Figure 7: Minima of $f(z) = e^z + \frac{z^3}{5}$

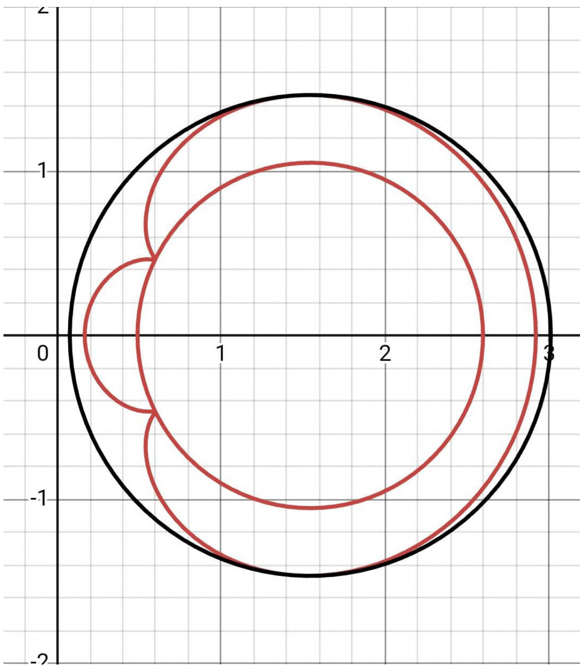


Figure 8: Radius of $f(z) = e^z + \frac{z^3}{5}$

3.3 Conclusion

We consider a complex function $f(z) = e^z + \frac{z^3}{5}$ associated with a sum of exponential and polynomial function and study the image of the mapping. While the information we obtain about the function from its image is limited, creative applications of computational tools such as *ComplexTool*, *Wolfram Mathematica* and *Desmos Online Graphing Calculator* yield a host of interesting results.

Chapter - 4

Function associated with Epitrochoid domain

4.1 Introduction

The epitrochoid, a smooth curve generated by rolling one circle around the exterior of another, is one of the most intriguing geometric shapes in mathematics. When divided into distinct regions, each portion of the epitrochoid domain retains its own unique properties, while simultaneously presenting interesting contrasts. In this chapter, we will embark on a focused exploration of the function associated with right half of epitrochoid domain $f(z) = \dfrac{z^2}{2} + z + 1$

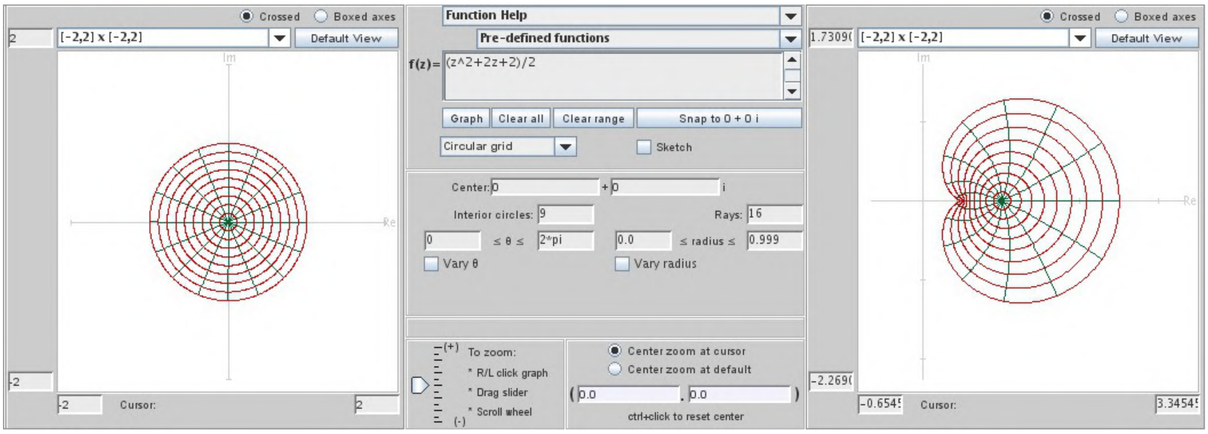


Figure 1: Image of unit disk under the function $f(z) = \dfrac{z^2}{2} + z + 1$

- f is univalent in D as the image curve does not have any intersecting. Also, $Re(f) > 0$.
- f(0) = 1 and f is starlike w.r.t f(0) = 1.
- f is symmetric about the real axis.
- f'(z) = z + 1. Hence, f'(0) = 1 > 0.

4.2 Main results

Theorem 4.1 The function $f(z) = \dfrac{z^2}{2} + z + 1$ maps D onto the region which is symmetric about the real axis lies completely in the right-half plane.

Proof. u(t) + iv(t) = f(e^{it}) = $\frac{e^{2it}}{2}$ + e^{it} + 1

$$= \frac{\cos(2t)}{2} + cost + 1 + i(\frac{\sin(2t)}{2} + sint)$$

gives

$$u(t) = \frac{\cos(2t)}{2} + cost + 1 \quad and \quad v(t) = \frac{\sin(2t)}{2} + sint$$

For z = e^{it}, the parametric equations of $f(z) = \dfrac{z^2}{2} + z + 1$ are

$$u(t) = \frac{\cos(2t)}{2} + cost + 1$$

and

$$v(t) = \frac{\sin(2t)}{2} + sint,$$

$-\pi < t < \pi$.

$$(u-1)^2 + v^2 = \frac{\cos^2 2t + \sin^2 2t}{4} + \cos^2 t + \sin^2 t + \cos 2t \cos t + \sin 2t \sin t$$

$$= \frac{5}{4} + (1 - 2\sin^2 t)\cos t + 2\sin^2 t \cos t$$

$$(u-1) + v^2 = \frac{5}{4} + \cos t$$

Now,

$$u = \frac{\cos 2t}{2} + \cos t + 1$$

$$u = \left(\cos t + \frac{1}{2}\right)^2 + \frac{1}{4}$$

$$\Rightarrow (v - \frac{1}{4})^2 = \left(\cos t + \frac{1}{2}\right)^2$$

$$\Rightarrow \cos t = -\frac{1}{2} \pm \sqrt{v - \frac{1}{4}}$$

$$\therefore (u-1)^2 + v^2 = \frac{5}{4} - \frac{1}{2} + \sqrt{v - \frac{1}{4}}$$

$$(u-1)^2 + v^2 = \frac{3}{4} + \sqrt{v - \frac{1}{4}}$$

Theorem 4.2 Let $a = 1.5$. *Let* r_a and R_a be given by

$$r_a = 1$$

and

$$R_a = 2$$

Then

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subseteq f(d) \subseteq \{w \in \mathbb{C} : |w - a| < R_a\}.$$

Proof. For z $= e^{it}$, the parametric equations of $f(z) = \dfrac{z^2}{2} + z + 1$ are

$$u(t) = \frac{\cos(2t)}{2} + \cos(t) + 1$$

and

$$v(t) = \frac{\sin(2t)}{2} + \sin(t)$$

$-\pi < 0 < \pi$

Evaluating $u(t)$ at $t = 0$ and $t = \pi$:

$$u(0) = \frac{5}{2}, \quad u(\pi) = \frac{1}{2}$$

Given $a = 1.5$, the function $z(t)$ becomes:

$$z(t) = \left(\frac{\cos(2t)}{2} + \cos(t) + 1 - a\right)^2 + \left(\frac{\sin(2t)}{2} + \sin(t)\right)^2$$

The derivative $z'(t)$ was plotted for t in the interval $[0, 2\pi]$. The critical points when $z'(t) = 0$ solving this in Mathematica are:

$$t = 0, \quad t = 1.5708, \quad t = 3.14159 \approx \pi, \quad t = 4.71239, \quad t = 6.28319$$

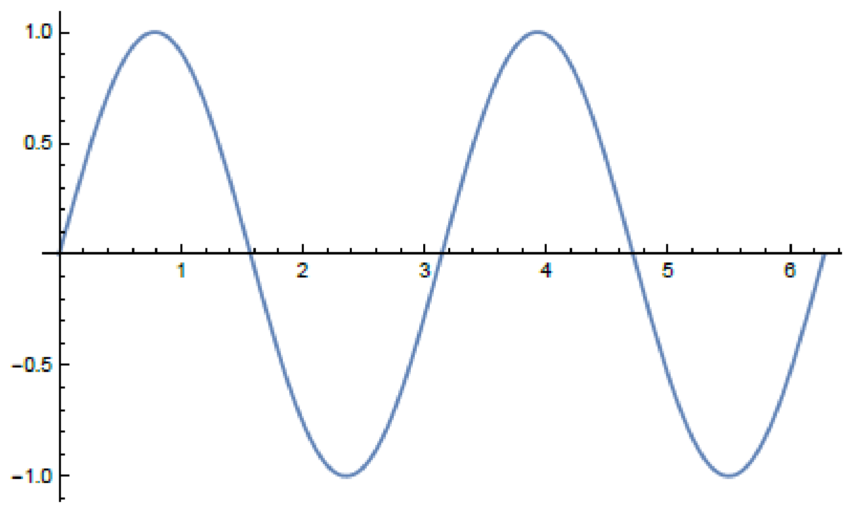


Figure 2: *Figure 2 : Image of* $z'(t)$

To check the nature of the critical points:

$$z''(0) = 2 \quad (point of minima)$$
$$z(0) = 1 \quad (minima)$$
$$z''(1.5707963267948966) = -2 \quad (point of maxima)$$
$$z(1.5707963267948966) = 2 \quad (maxima)$$

For $t = \pi$:

$$z''(\pi) = 2$$
$$z(\pi) = 1 \quad (same)$$

For $t = 4.71238898038469$:

$$z''(4.71238898038469) = -2$$
$$z(4.71238898038469) = 2 \quad (same)$$

For $t = 6.283185307179586$:

$$z''(6.283185307179586) = 2$$
$$z(6.283185307179586) = 1 \quad (same)$$

Hence, $r_a = 1$ and $R_a = 2$.

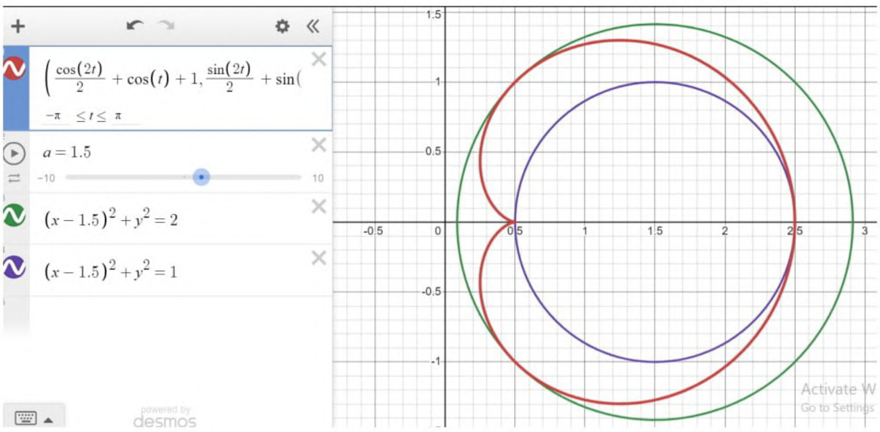


Figure 3 : *Calculations of* $z(t)$ *and plotting of functions when* $a = 1.5$

4.3 Conclusion

This chapter provided a detailed analysis of the cardioid function $f(z) = \frac{z^2}{2} + z + 1$ within the unit disk. We concluded that this function consists of several important properties: univalence, a positive real part, a starlike image symmetric about the real axis, and a positive derivative at the origin. These properties were proven through a combination of analytical techniques and computational process using Mathematica. The determination of inner and outer circular bounds for the image of the unit disk helps to offer a quantitative description of the function's mapping behavior.

Chapter 5

Function Associated with the Right Half of the Lemniscate Domain

5.1 Introduction

The **lemniscate**, a symmetrical curve resembling the infinity symbol (∞), stands as one of the most captivating mathematical shapes. When divided into two symmetrical halves, each portion of the lemniscate retains its own unique characteristics, while at the same time offering intriguing distinctions. In this chapter, we will embark on a focused exploration of the function associated with right half of Lemniscate domain, $\phi(z) = 1 + z + \sqrt{1+z}$

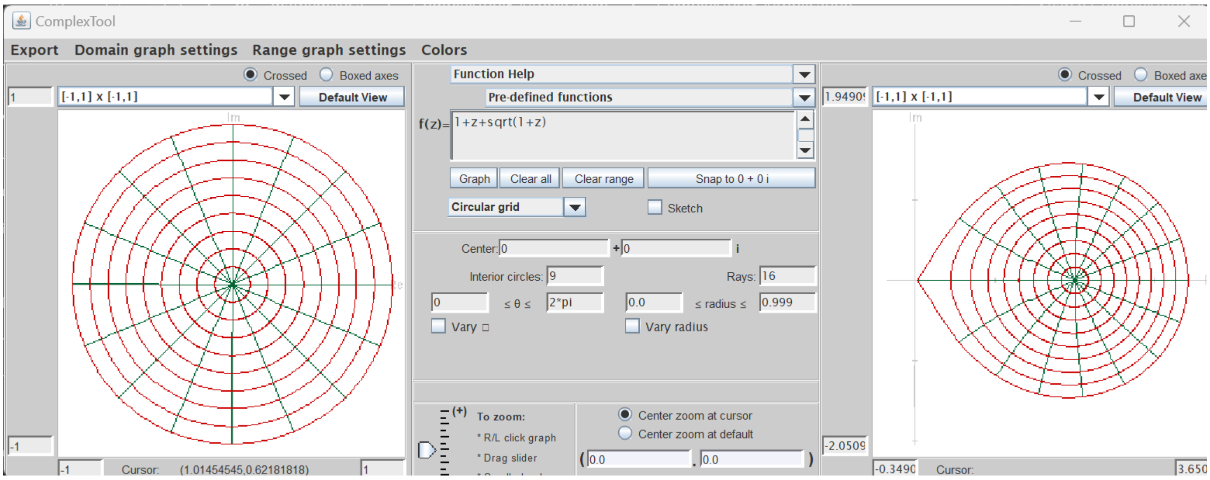


Figure 5.1: Image of disk under the function $\phi(z)$

This function satisfies the following:-

1.$\phi(z)$ is univalent
2.$Re(\phi) > 0$
3.$\phi(\mathbb{D})$ is starlike with respect to $\phi(0) = 2$
4.$\phi(\mathbb{D})$ is symmetric about the real axis
5.$\phi'(0) > 0$

We now prove that ϕ satifies each of these properties

1. Univalence Condition

ϕ is univalent in $\mathbb{D}$ as it maps the domain onto non intersecting arcs and rays.(*see fig. 5.1*)

2. In order to prove that $\Re\{f(z)\} > 0$, $z \in \mathbb{D}$, it suffices to show that

$$\Re\left\{f(e^{it})\right\} \geq 0, \quad t \in [0, 2\pi).$$

Let $z = e^{it}$, $t \in [0, 2\pi)$, then

$$1+e^{it}+\sqrt{e^{it}+1} = \begin{cases} 1+\cos t + i\sin t + \sqrt{2\cos(t/2)}\,(\cos(t/4)+i\sin(t/4)), & \text{for } t \in [0,\pi), \\ 0 & \text{for } t = \pi, \\ 1+\cos t + i\sin t + \sqrt{2|\cos(t/2)|}\,(\cos(t/4)-i\sin(t/4)), & \text{for } t \in (\pi, 2\pi). \end{cases}$$

Now, some simple calculations show that $\Re\left\{1+e^{it}+\sqrt{e^{it}+1}\right\} = 0$, if and only if $t = \pi$, which implies that $\Re\{f(z)\} > 0$ in $\mathbb{D}$

3. To prove $\phi(\mathbb{D})$ is starlike with respect to $\phi(0) = 2$, we will show that

$$\Re\left(\frac{z\phi'(z)}{\phi(z)-2}\right) > 0$$

where we take $z = re^{i\theta}$.
Since $z \in \mathbb{D}$, on plotting $\Re\left(\frac{z\phi'(z)}{\phi(z)-2}\right)$ in Mathematica, we get

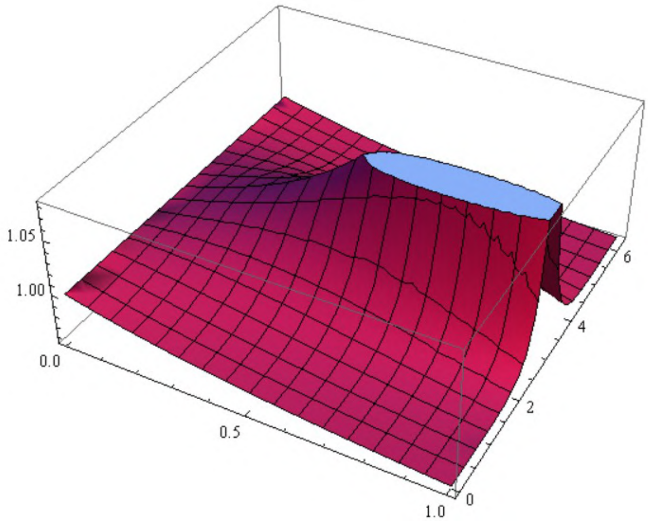


Figure 5.3: $\Re\left(\frac{z\phi'(z)}{\phi(z)-2}\right)$

Which shows that $\Re\left(\frac{z\phi'(z)}{\phi(z)-2}\right) > 0$

4. The graph of $\phi(z)$ (*see fig. 5.1*) shows that the image formed above and below the real axis are the same, therefore, we can say that $\phi(z)$ is symmetric about the real axis over D, i.e., $\phi(D)$ is symmetric about the real axis.

5.

$$\phi'(z) = 1 + \frac{1}{2\sqrt{1+z}}$$

and as

$$\phi'(0) = 1 + \frac{1}{2}$$

hence

$$\phi'(0) > 0$$

5.2 Main Results

Theorem 5.1 Let $a = 1$. Let r_a and R_a be given by

$$r_a = 0.816013$$

and

$$R_a = 1 + \sqrt{2}.$$

Then

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subseteq \phi(D) \subseteq \{w \in \mathbb{C} : |w - a| < R_a\}.$$

Proof. For $z = e^{it}$, the parametric equations of the function $\phi(z) = 1 + z + \sqrt{1+z}$ are

$$u(t) = 1 + \cos t + \sqrt{2\cos\frac{t}{2}}\left(\cos\frac{t}{4}\right), \quad v(t) = \sin t + \sqrt{2\cos\frac{t}{2}}\left(\sin\frac{t}{4}\right), \quad -\pi < t \leq \pi.$$

The square of the distance from the point $(1, 0)$ to the points on the function is given by

$$z(t) = (1 - u(t))^2 + (v(t))^2.$$

It can be easily seen by calculating through mathematica software that

$$z'(t) = \frac{1}{2}\left(-4\cos\left(\frac{t}{4}\right) - \frac{\sqrt{2}\left(3 + 4\cos\left(\frac{t}{2}\right) + 4\cos(t)\right)}{\sqrt{\cos\left(\frac{t}{2}\right)}}\right)\sin\left(\frac{t}{4}\right)$$

Next, we will plot $z'(t)$ to find the critical points.

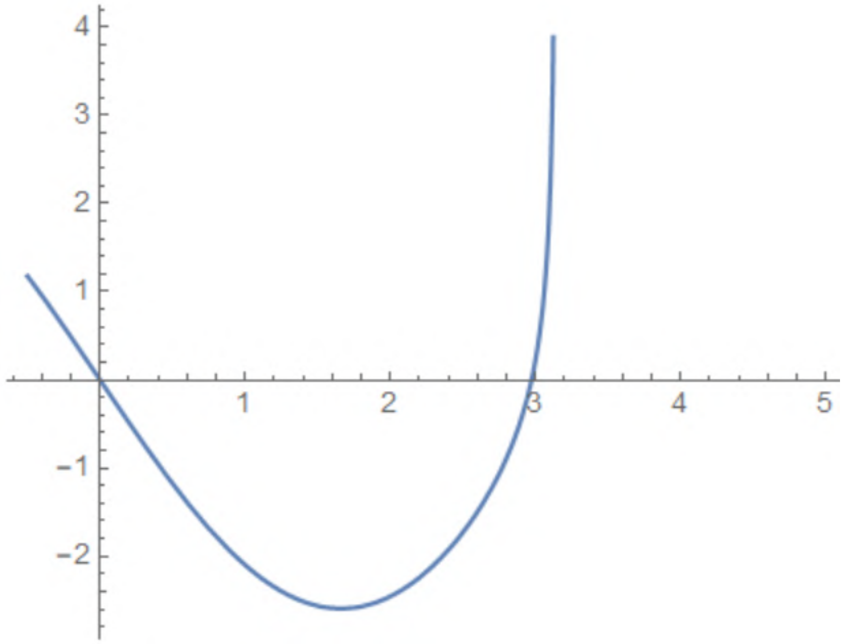


Figure 5.4: $f'(z)$

After solving this in Mathematica, we find two critical points at $t = 0$ and $t \approx 2.97251$.

At $t = 0$,

$$z''(0) \approx -2.44454 \quad (\leq 0,\ \text{point of maxima})$$

$$z(0) = (1+\sqrt{2})^2 \quad \text{(maxima)}$$

Hence, $R_a = Max\sqrt{z(t)} \approx 1 + \sqrt{2}$.

At $t \approx 2.97251$,

$$z''(2.97251) \approx 7.55781 \quad (\geq 0,\ \text{point of minima})$$

$$z(2.97251) \approx 0.66587 \quad \text{(minima)}$$

Thus, $r_a = min\sqrt{z(t)} \approx 0.81601$

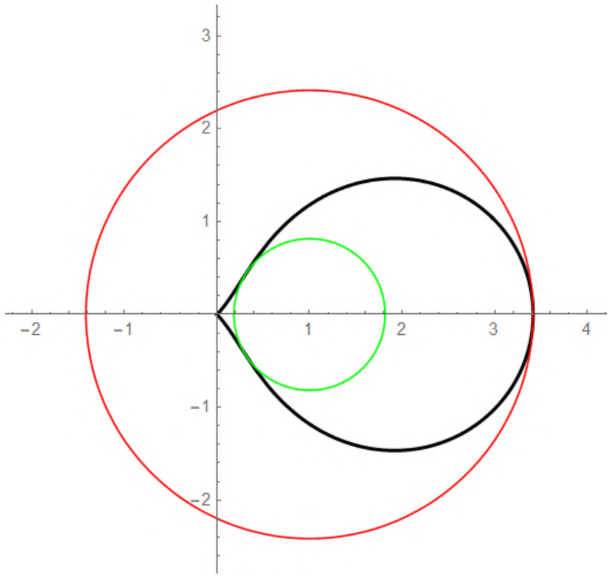


Figure 5.5: $a = 1$

Theorem 5.2 Let $a = \frac{\sqrt{2}}{2} + 1$.*Let* r_a and R_a be given by

$$r_a = 1.36996$$

and

$$R_a = 1 + \frac{1}{\sqrt{2}}.$$

Then

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subseteq \phi(D) \subseteq \{w \in \mathbb{C} : |w - a| < R_a\}.$$

Proof. For $z = e^{it}$, the parametric equations of the function $\phi(z) = 1 + z + \sqrt{1+z}$ are

$$u(t) = 1 + \cos t + \sqrt{2\cos\frac{t}{2}}\left(\cos\frac{t}{4}\right), \quad v(t) = \sin t + \sqrt{2\cos\frac{t}{2}}\left(\sin\frac{t}{4}\right), \quad -\pi < t \leq \pi.$$

The square of the distance from the point $(\frac{\sqrt{2}}{2}+1,0)$ to the points on the function is given by

$$z(t)=\left(\frac{\sqrt{2}}{2}+1-u(t)\right)^2+(v(t))^2.$$

It can be easily seen by calculating through mathematica software that

$$z'(t)=\frac{1}{2}\left(-4\cos\left(\frac{t}{4}\right)-\frac{\sqrt{2}\left(3+4\cos\left(\frac{t}{2}\right)+4\cos(t)\right)}{\sqrt{\cos\left(\frac{t}{2}\right)}}\right)\sin\left(\frac{t}{4}\right)$$

After solving this in Mathematica, we find two critical points at $t=0$ and $t\approx 2.5212404$.

At $t=0$,

$$z''(0)\approx -0.6553301 \quad (\leq 0,\ \text{point of maxima})$$

$$z(0)=\left(1-\frac{1}{\sqrt{2}}+\sqrt{2}\right)^2 \quad (\text{maxima})$$

Hence, $R_a = Max\sqrt{z(t)} \approx \left(1+\frac{1}{\sqrt{2}}\right)$.

At $t\approx 2.5212404$,

$$z''(2.5212404)\approx 1.5256894 \quad (\geq 0,\ \text{point of minima})$$

$$z(2.5212404)\approx 1.87678668 \quad (\text{minima})$$

Thus, $r_a = min\sqrt{z(t)} \approx 1.36996$

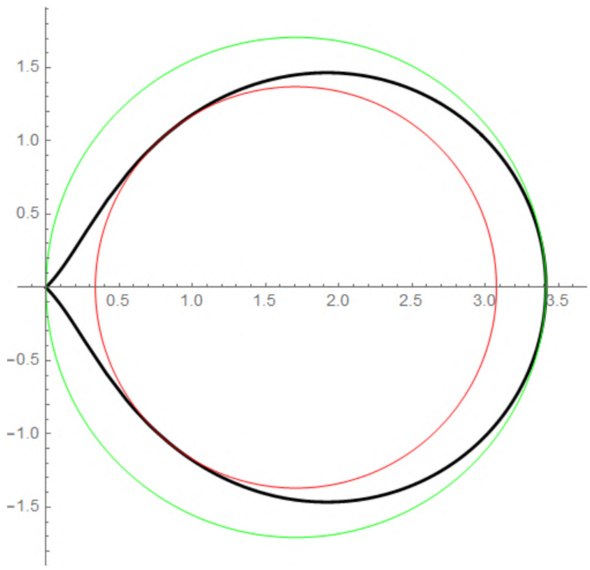


Figure 5.7: $a=\frac{\sqrt{2}}{2}+1$

5.3 Conclusion

This chapter provided a detailed analysis of the function associated with right half of Lemniscate domain $\phi(z) = 1 + z + \sqrt{1+z}$ within the unit disk. We established that this function possesses several important properties: univalence, a positive real part, a starlike image symmetric about the real axis, and a positive derivative at the origin. These properties were proven through a combination of analytical techniques and computational verification using Mathematica. The determination of inner and outer circular bounds for the image of the unit disk offers a quantitative description of the function's mapping behavior.

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