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Project Report of 2025: SVP-2504

"Mathematical Modelling and Optimal Control for Protecting Crop Productivity in a Warming Climate"

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<u>SRIVIPRA PROJECT 2025</u>

Title : Mathematical Modelling and Optimal Control for Protecting Crop Productivity in a Warming Climate

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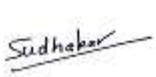

Signature of Mentor

Certificate of Originality

This is to certify that the aforementioned students from Sri Venkateswara College have participated in the summer project SVP-2504 titled “**Mathematical Modelling and Optimal Control for Protecting Crop Productivity in a Warming Climate”**.The participants have carried out the research project work under my guidance and supervision from 1st July, 2025 to 30th September 2025. The work carried out is original and carried out in an online/offline/hybrid mode.

Signature of Mentor

Acknowledgements

We would like to express our sincere gratitude to the **Department of Mathematics, Sri Venkateswara College, University of Delhi**, for providing us with the opportunity to undertake this research project.

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Lastly, we express our gratitude to the college for nurturing a culture of research and learning, and for giving budding mathematicians an opportunity to explore, innovate, and grow.

Regards
Mr. Rohan Mishra

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Chapter 1

Introduction

1.1 Background of Climate Change and Agriculture

Agriculture the foundation of human civilization from the beginning of permanent human settlement.It is the backbone of the human life from thousand of years.Over last few decades the Crop production has been recorded with a high growth rate because of advancement in technology i.e. in irrigation,machines,high yielding varieties and major contributor to it is the synthetic fertilizer which become popular with green revolution. But gradually the trend is shifting, modern agriculture is very sensitive to climate change today.

Numerous studies have proven that the crop yield is strongly effected by the temperature, precipitation, and the concentration of CO_2 in atmosphere.Studies have been demonstrated that the crop production have already been negatively impacted by the global warming [1] and the crop productivity in the U.S. like countries also fall sharply beyond the optimum temperature threshold.This supports the inclusion of temperature-damage term in the productivity equation in the model [2]. The use of synthetic nitrogen fertilizers have enhanced the crop production to a large extent by fulfilling the nutrient limitations. Yet, the efficiency declines as fertilizer application rate increases leading to diminishing returns because of environmental losses like leaching and nitrous oxide emission.The motivates the inclusion of fertilizer stock equation and diminishing return terms in our model[3]. Agriculture is a significance contributor to the global green house gases i.e. NO_2 from fertilizers and CO_2 from the land use changes. Integrated Assessment Model have been worked on the emissions, economic activity and climate change [4, 5].

Using this idea fertilizers is treated as a driver of crop growth and source of emission that increases temperature in our model. These all together forms the back ground of the model.

1.2 Impact of Warming Climate on Crop Productivity

Global warming s the phenomenon of global temperature rise. This is induced by the greenhouse gases which traps heat radiations and that leads to rise in the atmospheric temperature. These gases are emitted by the excessive anthropogenic activities. Mathematical models have been used to examine the impact of global rise in CO_2 concentration and surface temperature rise on the crop yield. The studies have been revealed that the elevated CO_2 leads to decline in crop equilibrium[6]. Higher CO_2 can enhance the photosynthesis and water-use efficiency, often called as CO_2 fertilization effect.But beyond the threshold the the negative impacts of rising temperature makes these benefits void [6]. Climatic change often leads to frequent extreme weather events that damages the crop yield, reduces soul fertility, increases prate of pest infection and disturbs the planting cycles[7]. As global warming continues the cumulative stress on the soil, water table, and crop physiology reduces the crop yield.This model couple the CO_2, temperature, and water availability to analyses and predicts the net decline in the productivity.

1.3 Previous of Literature Works

There were many previous works which demonstrates the dynamics of productivity of crop, increased CO_2, global warming, fertilizer use etc. here, some such previous works are listed which motivated our model.

WD Nordhaus had created an Integrated Assessment model (IAM) that connects greenhouse gas emissions, climate change, and the global economy. Its objective is to examine the best climate policy and economic trade-offs, paying particular attention to how rising temperatures affect global production, particularly agriculture [4, 5].

[8] developed a Environmental Policy Integrated Climate (EPIC) model which demonstrates the crop growth, soil water level, nutrient cycling etc. It's notion is to demonstrates the impacts of the climate variability, fertilizer use, and soil processes on agricultural productivity.

[9] had developed a Decision Support System for Agrotechnology Transfer (DSSAT) is a widely used model for crop simulation. It's objective is to predict crop yields under different climatic conditions, soil conditions, and strategies to use fertilizer.

[10] designed a Integrated Model to Assess the Global Environment (IMAGE) connects agriculture, land use, energy, and climate. Its goals include assessing the role that agriculture—including the use of fertilizer—plays in climate change and researching solutions to mitigate its effects for sustainable food and climate systems.

[11] is another dynamic global crop and vegetation model. It's goal is to relate food production to the global carbon and water cycles while simulating global agricultural yields under climate change, CO_2 fertilization, and water/nutrient restrictions, including fertilizer management.

1.4 Objective of the Model

The Model demonstrates the dynamics between the Productivity of crop, Global warming(temperature anomaly), Plant available soil Nitrogen dynamics and fertilizer application rate(control parameter).

The Objectives of our model;

1. To understand the interaction between the agriculture and climate
2. To analyses the feedback mechanism i.e. the changes to one quantity with change in other
3. To Identify the Equilibrium states and to analyze the stability conditions
4. To access and plan for sustainability
5. To formulate policy and plan for future as per their mutual behavior

Chapter 2

Model Formulation

2.1 The System of Equations for the Model

The model involves the state Variables as $P(t)$,$T(t)$ and $N(t)$ representing the crop productivity, temperature anomaly of global warming and plant available soil nitrogen respectively.

The dynamics of the Crop production$P(t)$ at any time t represent the the total crop yield in that particular time.let α be the intrinsic growth rate of the crop production which represents the natural growth potential that crops possess under optimal conditions[12]. The losses of yield also takes place because of the negative impact of global warming.Let β be the sensitivity of crop productivity because of temperature.Thus, $-\beta TP$ minimizes the crop productivity[1]. Again, the crop productivity is also altered by the effect of soil nitrogen which enhances the crop productivity.let γ_N be the effectiveness of soil nitrogen which enhances the crop yield and $\gamma_N N$ represents the response of crop to nitrogen availability[13]. Let δ_N be the diminishing return coefficient for excessive nitrogen.And $-\delta_N N^2$ is the quadratic penalty term usually used in the agronomic yield-fertilizer response function[14]. Let $u(t)$ represents the control parameter which represents irrigation, pest control,soil management or harvest intensity.The Eq. (1) represents the dynamics of Crop productivity;

$$\frac{dP}{dt} = (\alpha - \beta T + \gamma_N N)P - \delta_N N^2 - u(t) \tag{1}$$

Considering the dynamics of global warming$G(t)$,let ρ be the conversion factor from global greenhouse gas forcing into temperature anomaly.Assume E_0 be the baseline emission from the non agricultural sources [15]. And let ϵ_F

be the emission factor per unit of fertilizer applied in the crop field and $\epsilon_F F$ is the emission from fertilizer application [4]. The greenhouse forcing also takes place because of the emission from land use change and agricultural production i.e. $[\epsilon_P \phi_P P]$. Let ϵ_P and ϕ_P are the emission factor per unit crop production and scaling factor(to convert productivity P to area/emission intensity) respectively [16]. The heat is also been absorbed by the natural means so, κ be the climate damping rate represents natural dissipation or uptake of heat by natural sinks like oceans and atmosphere[5]. Hence, Eq.(2) represents the dynamics of Global Warming;

$$\frac{dT}{dt} = \rho(E_0 + \epsilon_F F(t) + \epsilon_P \phi_p P) - \kappa T \tag{2}$$

Considering the plant available Soil Nitrogen $N(t)$, which provides nutrients to the plant. Let F be the external nitrogen applied as synthetic fertilizer. Consider, η_P be the nitrogen uptake by the crop per unit of productivity so, $-\eta_P P$ is the factor responsible for it[17]. Further, $-\ell N$ represents environmental nitrogen losses.[18] Hence,Eq.(3) represents the dynamics of Soil nitrogen;

$$\frac{dN}{dt} = F(t) - \eta_P N - \ell N \tag{3}$$

The fertilizer application rate,F(t) is a control dynamic not a state variable.

The System of Model is consists of three non-linear differential equations.The (4) represents the equations of the model.

$$\begin{cases} \frac{dP}{dt} = (\alpha - \beta T + \gamma_N N)P - \delta_N N^2 - u(t) \\ \frac{dT}{dt} = \rho(E_0 + \epsilon_F F(t) + \epsilon_P \phi_p P) - \kappa T \\ \frac{dN}{dt} = F(t) - \eta_P N - \ell N \\ \end{cases} \tag{4}$$

Where, $P(t) \geq 0, T(t) \geq 0$ and $N(t) \geq 0$

2.2 Schematic representation of the Model

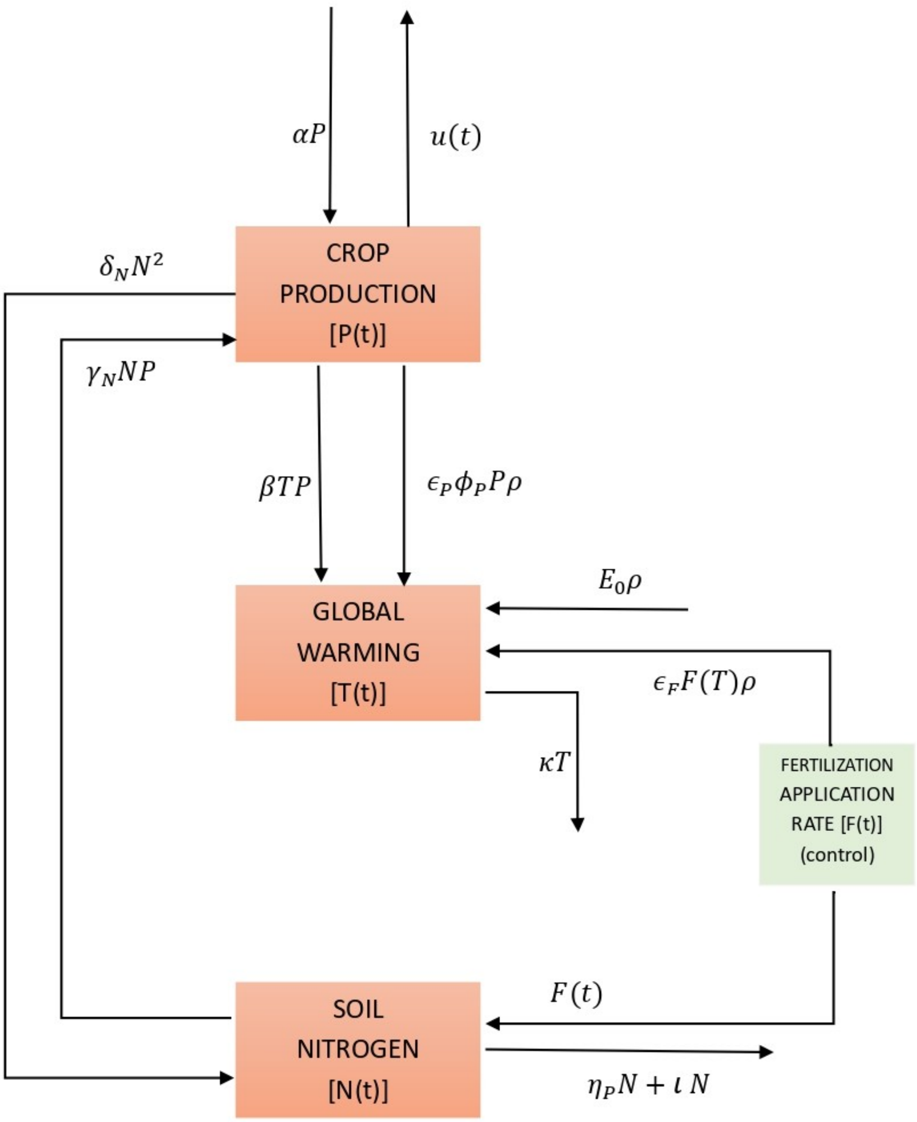

Figure 2.1: Schematic representation of the relationship between crop productivity, temperature anomaly of global warming and plant available soil nitrogen and relation with the control parameter of fertilizer application rate

2.3 Description of State Variables, Parameters and Controls involved in the model

Table 2.1: State variables and their description

Symbol	Description	Units
$P(t)$	Crop productivity (yield index or tons per hectare)	tons/ha
$T(t)$	Temperature anomaly (global warming relative to baseline)	°C
$N(t)$	Plant-available soil nitrogen stock	kg N/ha

Table 2.2: Model parameters and their description

Symbol	Description	Units
α	Intrinsic crop productivity growth rate	per time
β	Sensitivity of productivity to temperature stress	per (°C·time)
γ_N	Effectiveness of soil nitrogen in enhancing productivity	per (kg N·time)
δ_N	Crop damage / diminishing returns from excessive nitrogen	per (kg^2 N^2·time)
ρ	Conversion factor from emissions to temperature change	°C per emission unit per time
E_0	Baseline non-agricultural emissions	emission units per time
ϵ_F	Emission factor of fertilizer use (N_2O per kg N applied)	emission units per kg N
ϵ_P	Emission factor linked to crop production (land use)	emission units per yield
ϕ_P	Scaling factor linking productivity to emissions	dimensionless
κ	Climate damping / heat dissipation rate	per time
η_P	Nitrogen uptake by crops per unit productivity	kg N per yield per time
ℓ	Soil nitrogen loss rate (leaching, volatilization, denitrification)	per time

Table 2.3: Model controls and their description

Symbol	Description	Units
$F(t)$	Fertilizer application rate	kg N/ha per time
$u(t)$	Management/harvest/adaptation effort (irrigation, pest control, resource cost)	cost units per time

Chapter 3

Analysis of the Model

3.1 Boundedness Analysis

We used standard comparison theorem to prove boundedness and positivity of the model solution as it is done by [19]. We can easily show that that in the region of attraction, which is an invariant region that attracts all solutions of the system (4).

Theorem 3.1.1 (Boundedness of Solutions)**.** *:*
Consider the system

$$\frac{dP}{dt} = (\alpha - \beta T + \gamma_N N)\, P - \delta_N N^2 - u(t), \tag{1}$$

$$\frac{dT}{dt} = \rho\,(E_0 + \epsilon_F F(t) + \epsilon_P \phi_P P) - \kappa T, \tag{2}$$

$$\frac{dN}{dt} = F(t) - \eta_P P - \ell N, \tag{3}$$

with non-negative initial conditions $P(0), T(0), N(0) \geq 0$*. Assume that the control variables are bounded as*

$$0 \leq F(t) \leq F_{\max}, \qquad 0 \leq u(t) \leq u_{\max}, \qquad \forall t \geq 0,$$

and all parameters are nonnegative with $\kappa, \ell > 0$*. Then every solution* $(P(t), T(t), N(t))$ *of (1),(2) and (3) remains non- negative and ultimately bounded. Specifically,*

$$0 \leq N(t) \leq N_m := \frac{F_{\max}}{\ell}, \quad 0 \leq P(t) \leq P_m, \quad 0 \leq T(t) \leq T_m, \tag{5}$$

for all sufficiently large t*, where* P_m *and* T_m *are finite positive constants depending on system parameters and initial data.*

Proof. :

Positivity: As, $F(t), u(t) \geq 0$ and parameters are non-negative, the vector field points inward on the boundary of the positive orthant. By the standard positivity lemma for ordinary differential systems, solutions with non-negative initial data remain nonnegative for all $t \geq 0$.

Boundedness of $N(t)$:
Now, From (3),

$$\frac{dN}{dt} = F(t) - \eta_P P - \ell N \leq F(t) - \ell N \leq F_{\max} - \ell N$$

Consider the scalar ordinary differential equation,

$$\frac{dY}{dt} = F_{\max} - \ell Y, \quad Y(0) = N(0), \tag{6}$$

which has the solution

$$Y(t) = \frac{F_{\max}}{\ell} + \left(N(0) - \frac{F_{\max}}{\ell} \right) e^{-\ell t} \tag{7}$$

By the comparison theorem [20], $N(t) \leq Y(t)$ for all $t \geq 0$. Hence,

$$\limsup_{t \to \infty} N(t) \leq \frac{F_{\max}}{\ell} = N_m \tag{8}$$

Boundedness of $P(t)$:
Again From (1), using $N(t) \leq N_m$,

$$\frac{dP}{dt} \leq (\alpha + \gamma_N N_m) P \tag{9}$$

Thus, $P(t)$ grows exponentially. As, $\delta_N N^2$ and $u(t)$ provide additional negative feedback, $P(t)$ is uniformly bounded above by any constant P_m depending on initial data and parameters.
This implies that $P(t)$ cannot grow faster than an exponential function of time. However, equation (1) also contains negative terms: $-\delta_N N^2$ and $-u(t)$. These terms shows damping effects which reduce growth when N is large or when $u(t)$ is applied.
Therefore, the growth of $P(t)$ is restricted between the positive term $(\alpha - \beta T + \gamma_N N)P$ and the negative feedbacks $-\delta_N N^2$ and $-u(t)$. This ensures

that $P(t)$ is ultimately bounded above by some finite constant $P_m > 0$ which depends on the system parameters and the initial data.
so, that there exists a constant P_m such that

$$0 \leq P(t) \leq P_m, \qquad \forall t \geq 0. \tag{10}$$

Boundedness of $T(t)$:
Again From (2), and bounds $F(t) \leq F_{\max}$, $P(t) \leq P_m$,

$$\frac{dT}{dt} \leq \rho \left(E_0 + \epsilon_F F_{\max} + \epsilon_P \phi_P P_m\right) - \kappa T. \tag{11}$$

Consider the scalar ordinary differential equation

$$\frac{dW}{dt} = \rho \left(E_0 + \epsilon_F F_{\max} + \epsilon_P \phi_P P_m\right) - \kappa W, \tag{12}$$

which has equilibrium

$$W^* = \frac{\rho}{\kappa} \left(E_0 + \epsilon_F F_{\max} + \epsilon_P \phi_P P_m\right). \tag{13}$$

By comparison, $T(t) \leq W(t)$ and hence

$$\limsup_{t \to \infty} T(t) \leq W^* = T_m. \tag{14}$$

All state variables $P(t), T(t), N(t)$ remain non-negative and are ultimately bounded above by finite constants P_m, T_m, N_m.This shows the boundedness of the system. □

3.2 Equilibrium Analysis

To determine the equilibrium points of the system, we set

$$\frac{dP}{dt} = \frac{dT}{dt} = \frac{dN}{dt} = 0. \tag{15}$$

System of equations

$$\frac{dP}{dt} = \left(\alpha - \beta T + \gamma_N N\right) P - \delta_N N^2 - u, \tag{1}$$
$$\frac{dT}{dt} = \rho \left(E_0 + \epsilon_F F + \epsilon_P \phi_P P\right) - \kappa T, \tag{2}$$
$$\frac{dN}{dt} = F - \eta_P P - \ell N. \tag{3}$$

Soil nitrogen equilibrium

From equation (3), we obtain

$$0 = F - \eta_P P^* - \ell N^* \quad \Rightarrow \quad N^* = \frac{F - \eta_P P^*}{\ell} \tag{16}$$

Temperature equilibrium

From equation (2), we obtain

$$0 = \rho\,(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*) - \kappa T^*,$$

which gives

$$T^* = \frac{\rho}{\kappa}\,(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*). \tag{17}$$

Productivity equilibrium

From equation (1), we have

$$0 = (\alpha - \beta T^* + \gamma_N N^*)P^* - \delta_N (N^*)^2 - u.$$

Substituting (16) and (17) into this expression yields

$$\left(\alpha - \beta\frac{\rho}{\kappa}\big(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*\big) + \gamma_N \frac{F - \eta_P P^*}{\ell}\right)P^* - \delta_N \Big(\frac{F - \eta_P P^*}{\ell}\Big)^2 - u = 0. \tag{18}$$

Eq. (17) provides a non-linear condition for the equilibrium crop productivity P^*.

Equilibrium points

(i) Trivial equilibrium

If $P^* = 0$, then from (16),(17) we obtain

$$(P^*, T^*, N^*) = \left(0,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F),\ \frac{F}{\ell}\right) \tag{19}$$

(ii) Positive equilibrium (sustainable crop)

If $P^* > 0$, then P^* must satisfy equation (18). The corresponding equilibrium point is

$$(P^*, T^*, N^*) = \left(P^*,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*),\ \frac{F - \eta_P P^*}{\ell}\right) \tag{20}$$

where, P^* is any positive solution of (18).

3.3 Existence of Equilibrium Points

In order to analyse the quantitative behavior of the system (4),concept of stability theory is applied as the system (4) is non-linear.
To analyze the existence of the equilibrium points obtained previously in (19) and (20).

Trivial equilibrium

The trivial equilibrium is given by

$$E_0 = \left(0,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F),\ \frac{F}{\ell}\right) \tag{21}$$

For $P^* = 0$, we have

$$N^* = \frac{F}{\ell} \geq 0 \quad \text{since } F \geq 0, \tag{22}$$

and

$$T^* = \frac{\rho}{\kappa}(E_0 + \epsilon_F F) \geq 0 \quad \text{since, all parameters are nonnegative.} \tag{23}$$

Hence, the trivial equilibrium always exists for non-negative parameter values.

The trivial equilibrium E_0 always exists for $F(t) \geq 0$.

Positive equilibrium

The positive equilibrium is given by

$$E^* = \left(P^*,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*),\ \frac{F - \eta_P P^*}{\ell}\right), \tag{24}$$

where P^* satisfies the nonlinear algebraic equation

$$\left(\alpha - \beta\frac{\rho}{\kappa}\big(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*\big) + \gamma_N \frac{F - \eta_P P^*}{\ell}\right) P^* - \delta_N \Big(\frac{F - \eta_P P^*}{\ell}\Big)^2 - u = 0. \tag{25}$$

Conditions for existence

1. From the nitrogen balance, we require

$$N^* = \frac{F - \eta_P P^*}{\ell} \geq 0 \quad \Rightarrow \quad P^* \leq \frac{F}{\eta_P} \tag{26}$$

Thus, any positive equilibrium must satisfy $0 < P^* < \frac{F}{\eta_P}$.

2. From the temperature balance,

$$T^* = \frac{\rho}{\kappa}(E_0 + \epsilon_F F + \epsilon_P \phi_P P^*) \geq 0 \tag{27}$$

which always holds since parameters are non-negative.

3. From the productivity equation (1), the algebraic condition (25) must have at least one positive solution for P^* in the admissible interval $0 < P^* < \frac{F}{\eta_P}$.

A positive equilibrium E^* exists if equation (25) satisfy at least one positive solution P^* such that $0 < P^* < \frac{F(t)}{\eta_P}$. This requires that the positive crop growth effects (α, γ_N) are sufficiently strong relative to the losses (δ_N, u) and the negative warming effect (β).

3.4 Local Stability Analysis

To study the local stability for the system, we linearize around the equilibrium points obtained earlier.

Jacobian matrix

Let f_1, f_2, f_3 denotes the RHS of equations (1),(2) and (3). The Jacobian matrix of the system is defined as

$$J(P,T,N) = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial N} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial T} & \frac{\partial f_2}{\partial N} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial T} & \frac{\partial f_3}{\partial N} \end{bmatrix}. \tag{28}$$

Computing the derivatives, we obtain

$$J(P,T,N) = \begin{bmatrix} \alpha - \beta T + \gamma_N N & -\beta P & \gamma_N P - 2\delta_N N \\ \rho \epsilon_P \phi_P & -\kappa & 0 \\ -\eta_P & 0 & -\ell \end{bmatrix}. \tag{29}$$

Stability of E_0(trivial equilibrium)

The trivial equilibrium is

$$E_0 = \left(0,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F),\ \frac{F}{\ell}\right).$$

At this equilibrium, the Jacobian becomes

$$J(E_0) = \begin{bmatrix} \alpha - \beta\frac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\frac{F}{\ell} & 0 & -2\delta_N\frac{F}{\ell} \\ \rho\epsilon_P\phi_P & -\kappa & 0 \\ -\eta_P & 0 & -\ell \end{bmatrix} \tag{30}$$

The eigenvalues should satisfy the characteristic equation

$$(\lambda + \kappa)(\lambda + \ell)\Big(\lambda - \big[\alpha - \beta\tfrac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\tfrac{F}{\ell}\big]\Big) = 0 \tag{31}$$

The eigenvalues are ;

$$\begin{cases} \lambda_1 = -\kappa < 0, \\ \lambda_2 = -\ell < 0, \\ \lambda_3 = \alpha - \beta\frac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\frac{F}{\ell} \end{cases} \tag{32}$$

Therefore:

If $\lambda_3 = \alpha - \beta\frac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\frac{F}{\ell} < 0$, then all eigenvalues are negative and the trivial equilibrium E_0 is locally asymptotically stable.

If $\lambda_3 = \alpha - \beta\frac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\frac{F}{\ell} > 0$, then E_0 is unstable.

The trivial equilibrium E_0 is always present. It is stable when crop growth is dominated by warming and nitrogen losses, i.e.

$$\alpha - \beta\frac{\rho}{\kappa}(E_0 + \epsilon_F F) + \gamma_N\frac{F}{\ell} < 0.$$

Stability of E^*(positive equilibrium)

The positive equilibrium is

$$E^* = \left(P^*,\ \frac{\rho}{\kappa}(E_0 + \epsilon_F F + \epsilon_P\phi_P P^*),\ \frac{F - \eta_P P^*}{\ell}\right), \tag{33}$$

where, P^* satisfies the nonlinear algebraic equation (25). At this point, the Jacobian is

$$J(E^*) = \begin{bmatrix} \alpha - \beta T^* + \gamma_N N^* & -\beta P^* & \gamma_N P^* - 2\delta_N N^* \\ \rho\epsilon_P \phi_P & -\kappa & 0 \\ -\eta_P & 0 & -\ell \end{bmatrix}. \tag{34}$$

The characteristic polynomial is

$$\lambda^3 + [\kappa + \ell - (\alpha - \beta T^* + \gamma_N N^*)]\lambda^2 +$$

$$[\kappa\ell - (\alpha - \beta T^* + \gamma_N N^*)(\kappa + \ell) + \beta P^* \rho\epsilon_P \phi_P + (\gamma_N P^* - 2\delta_N N^*)(-\eta_P)]\lambda +$$

$$[-(\alpha - \beta T^* + \gamma_N N^*)\kappa\ell + \beta P^* \rho\epsilon_P \phi_P \ell + (\gamma_N P^* - 2\delta_N N^*)(-\eta_P)(-\kappa)] = 0 \tag{35}$$

Or

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{36}$$

$$\begin{cases} a_1 &= \kappa + \ell - (\alpha - \beta T^* + \gamma_N N^*), \\ a_2 &= \kappa\ell - (\alpha - \beta T^* + \gamma_N N^*)(\kappa + \ell) + \beta P^* \rho\epsilon_P \phi_P + (\gamma_N P^* - 2\delta_N N^*)(-\eta_P), \\ a_3 &= -(\alpha - \beta T^* + \gamma_N N^*)\kappa\ell + \beta P^* \rho\epsilon_P \phi_P \ell + (\gamma_N P^* - 2\delta_N N^*)(-\eta_P)(-\kappa). \end{cases} \tag{37}$$

According to the Routh-Hurwitz stability criterion, the positive equilibrium E^* is locally asymptotically stable if

$$a_1 > 0, \qquad a_3 > 0, \qquad a_1 a_2 > a_3.$$

The positive equilibrium E^* exists if equation (25) admits a positive solution P^*, and it is locally stable if the Routh–Hurwitz conditions are satisfied.

3.5 Dynamic behavior of the Model

The dynamic behavior analysis of the non-linear ecological and crop production models has been evolved as a fundamental tool to understand the complex systems co-relation and how they evolved over the period of time under the influence of both external and internal drivers.Dynamic behavior analysis provides us a more broad idea about the system trajectories, reveals transient responses, convergence rates, oscillatory patterns and possible threshold points etc.

In the context of agriculture, climate change models the dynamic behavior analysis is relevant as crop productivity, soil nutrient availability, and climatic conditions are time-dependent and highly interlinked initially. Due to rise in temperatures, shifts in precipitation pattern, and intensified fertilizer usage creates feedback loops that stabilize, destabilize or fundamentally alter system outcomes. By analyzing the dynamics evolution the variables - crop production $P(t)$, temperature anomaly $t(t)$ and soil nitrogen $N(t)$. We can evaluate eventual equilibrium and also short term stress and adjustments which are necessary for the decision making.

This framework dynamic behavior analyzes used to go beyond the abstract form of mathematics to bridge the theory with the practical implication. it implies how sustainable crop yields depends on the interplay between global warming and fertilizer management discovers conditions under which soil quality have been degraded and classifies the stability of co-existence between the crop-climate-fertilizer system constituents.

It is the study combines mathematical tiger with graphical simulations by providing a comprehensive understanding of the long term and short term effect of climate and fertilizer dynamics or agricultural productivity.

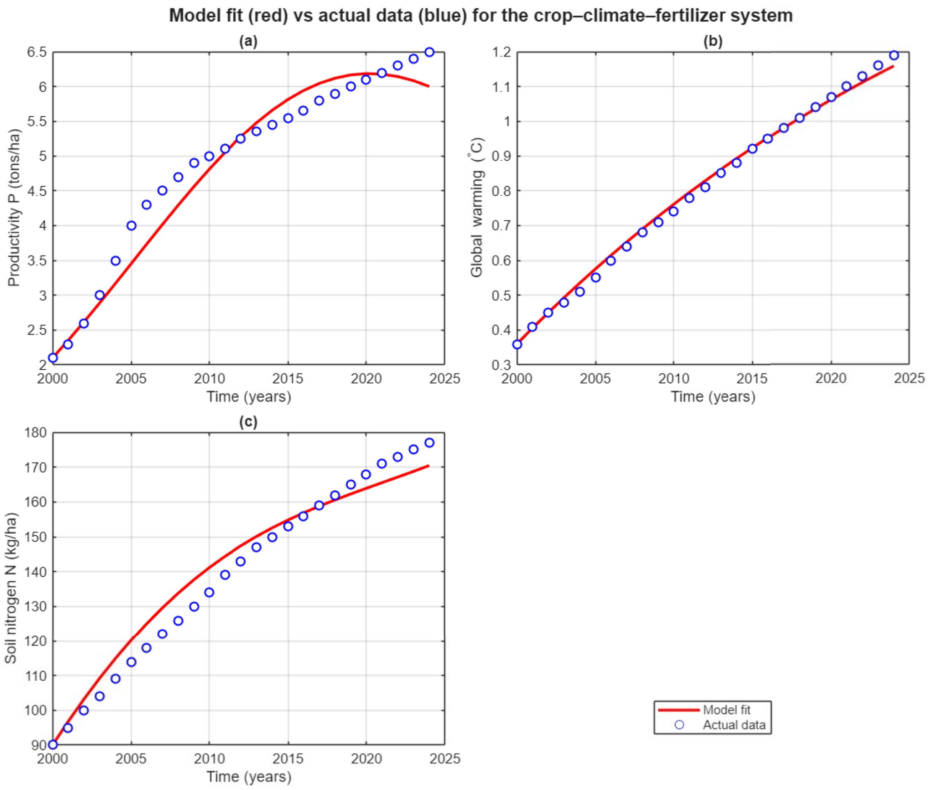

Figure 3.1: The Plotting of graph for the Model(With comparison with real data)

We used the MATLAB R2025a to plot the graph for this model. Here, we assigned some values to the parameters to analyze the behavior by plotting the graphs. So, the assigned values for the parameters are given in Table 3.1.We considered the $F(T), u(t)$ as constants as we avoided the optimal control analysis.

Symbol	Description	Initial guess	Lower bound	Upper bound
α	Crop productivity growth rate (per yr)	0.06	0.01	0.20
β	Sensitivity to temperature (°C^{-1}yr^{-1})	0.06	0.01	0.20
γ_N	Soil N effect on productivity (kgN^{-1}yr^{-1})	0.002	1×10^{-4}	0.010
δ_N	Crop damage from excess N (kg^{-2}yr^{-1})	1×10^{-5}	1×10^{-7}	5×10^{-5}
ρ	Emissions $\rightarrow$ temperature factor	0.010	0.001	0.050
ϵ_F	Fertilizer emission factor (kgN^{-1})	0.010	0.000	0.040
ϵ_P	Production emission factor	0.003	0.000	0.010
κ	Climate damping rate (per yr)	0.020	0.005	0.080
η_P	Nitrogen uptake (kgN per ton yield per yr)	20	1	80
ℓ	Soil N loss rate (per yr)	0.12	0.01	0.60
F_{const}	Fertilizer input (kgN/ha/yr)	120	0	400
u_{const}	Harvest/management (tons/ha/yr)	0.40	0	3.0
E_0	Baseline non-agricultural emissions	1.0	–	–
ϕ_P	Productivity $\rightarrow$ emissions scaling	1.0	–	–

Table 3.1: Assumed parameter values and bounds for the crop–climate–fertilizer model.

The results of the model–data comparison indicate that the proposed crop–climate–fertilizer system reproduces the real-world dynamics of productivity, soil nitrogen, and temperature anomalies with reasonable accuracy. Crop productivity shows strong agreement with the observed upward trend, though the model predicts a plateau and eventual decline after 2020, reflecting real-world risks of stagnation and collapse under climate stress, soil degradation, and unsustainable practices, as observed in South Asia and Sub-Saharan Africa. The negative trajectory here represents declining yields rather than unrealistic behavior. The temperature anomaly is captured with high fidelity, closely following the observed near-linear warming from about 0.35 °C in 2000 to over 1.1 °C in 2025, demonstrating robust climate feedback representation, though the flatness of the model trend may understate the intensity of actual warming. Soil nitrogen dynamics also align well, showing steady accumulation from fertilizer use, but the model slightly underestimates later-year values. This accumulation highlights the global challenge of fertilizer overuse, particularly evident in India and China, where rising applications have not produced proportional yield gains, leading to problems such as eutrophication and soil acidification. Overall, the curves demonstrate that while refinements in productivity and nitrogen parameterization are needed, the model effectively captures the coupled interactions between crop growth, climate change, and soil fertility, and underscores how climate stress and mismanagement of fertilizers can jointly push agriculture toward unsustainability.

3.5.1 Analysis of the system via time-series plot of the system

To plot the time-series analysis for the model we used MATLAB R2025a. We assumed the values of the parameters to plot the time-series curve and we treated the $F(t), u(t)$ controls as constant as we avoided the optimal control analysis.The Table 3.2 depicts the values for the parameters used to plot the time-series curve for the model.

Symbol	Description	Assumed Value	Unit
α	Intrinsic crop productivity growth rate	0.03	year^{-1}
β	Sensitivity of productivity to temperature stress	0.08	$(°\text{C}^{-1}\cdot\text{year}^{-1})$
γ_N	Effectiveness of soil nitrogen in enhancing productivity	0.002	$(\text{kg N})^{-1}\cdot\text{year}^{-1}$
δ_N	Crop damage / diminishing returns from excess nitrogen	1×10^{-5}	$(\text{kg N})^{-2}\cdot\text{year}^{-1}$
ρ	Conversion factor from emissions to temperature change	0.01	°C per emission unit $\cdot$ year^{-1}
E_0	Baseline non-agricultural emissions	1.0	emission units $\cdot$ year^{-1}
ϵ_F	Fertilizer emission factor (N_2O per kg N applied)	0.015	emission units $\cdot$ kg^{-1} N
ϵ_P	Production emission factor (land use related)	0.003	emission units $\cdot$ yield^{-1}
ϕ_P	Scaling factor linking productivity to emissions	1.0	dimensionless
κ	Climate damping / heat dissipation rate	0.02	year^{-1}
η_P	Nitrogen uptake by crops per unit productivity	25	kg N $\cdot$ $\text{yield}^{-1}\cdot\text{year}^{-1}$
ℓ	Soil nitrogen loss rate (leaching, volatilization, etc.)	0.1	year^{-1}
F	Fertilizer application rate	100	kg N $\cdot$ $\text{ha}^{-1}\cdot\text{year}^{-1}$
u	Management / harvest / adaptation cost	0.5	yield units $\cdot$ year^{-1}

Table 3.2: Assumed parameter values and bounds for the crop–climate–fertilizer model.

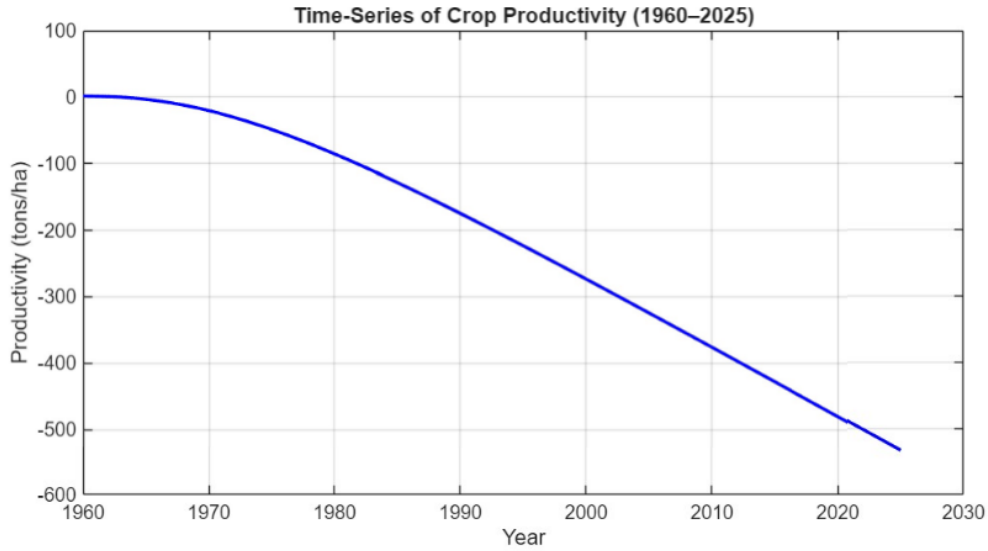

Figure 3.2: The time-series plot for the Crop productivity

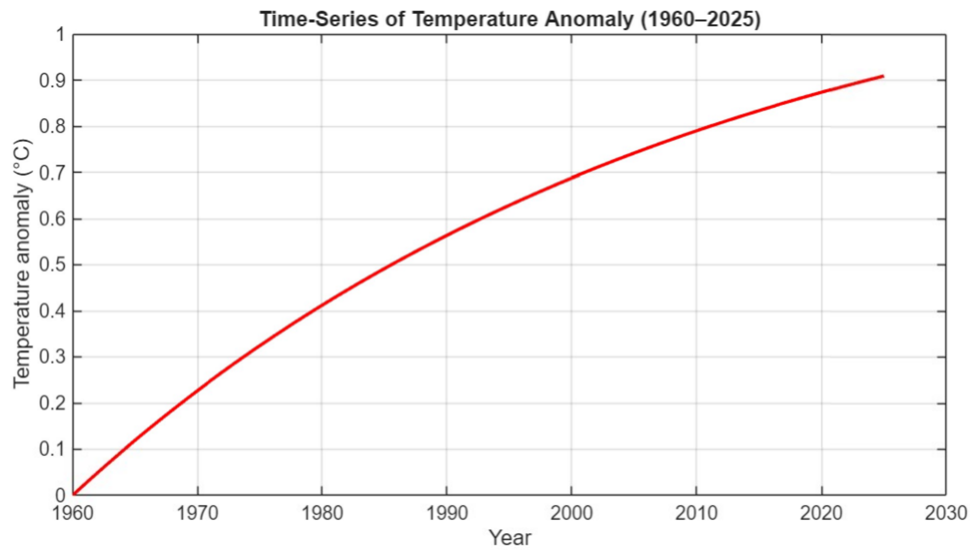

Figure 3.3: The time-series plot for Temperature

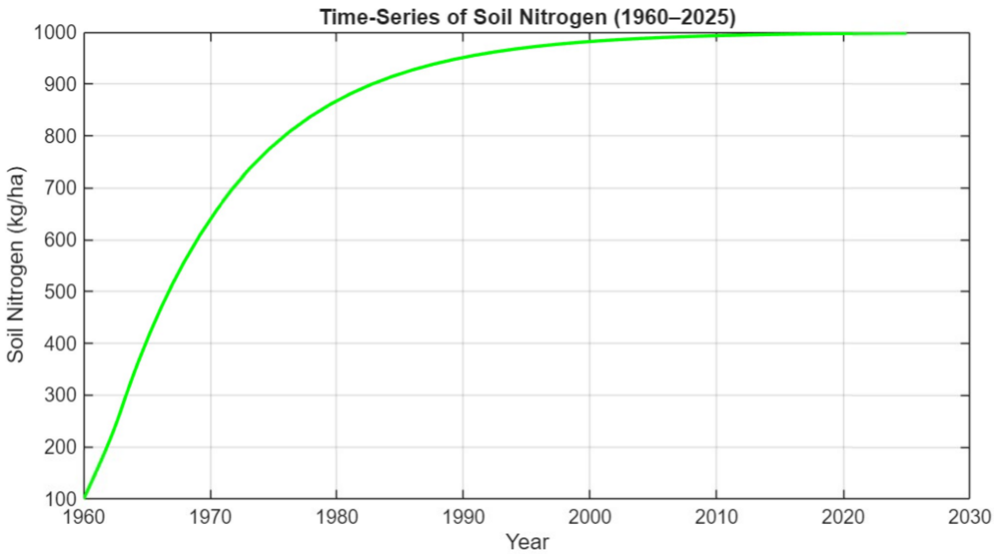

Figure 3.4: The time-series plot for Soil Nitrogen

The crop productivity curve shows an initial increase but later stagnates and eventually declines, suggesting that while technological inputs such as fertilizers and improved varieties initially boosted yields, these gains are increasingly offset by climate stress (rising temperatures) and soil degradation. At the same time, the soil nitrogen trajectory displays steady accumulation, highlighting the over-application of fertilizers without proportional crop uptake—a global issue, especially in regions like India and China, that contributes to eutrophication, soil acidification, and greenhouse gas emissions. The temperature anomaly curve, meanwhile, shows a near-linear rise that the model captures reasonably well, though in a smoother form than real-world data, meaning it may underplay extremes such as heatwaves and droughts. Taken together, these results emphasize the persistent climatic stress on agriculture, consistent with the 1.1 °C global warming observed since 1960, and illustrate how unsustainable fertilizer use and rising temperatures jointly threaten long-term crop productivity and environmental stability.

Chapter 4

Conclusion

The model of crop, climate, and fertilizer introduced in this research offers a dynamic system to investigate the relationship among agricultural output, soil quality, and climate change. The analysis indicates that crop yields cannot perpetually rise with more fertilizer; rather, overuse results in nitrogen saturation, deterioration of soil, and heightened emissions, while increasing temperatures add further stress that reduces productivity. The analysis of boundedness and equilibrium indicates the presence of both trivial (collapse) and positive (sustainable) equilibria, where stability is determined by the interplay between crop growth factors and climate stressors. Numerical simulations reveal a plausible path where productivity reaches its peak and subsequently falls, temperature anomalies progressively increase, and nitrogen builds up past sustainable thresholds—trends that align with observed agricultural patterns in areas like South Asia and Sub-Saharan Africa.

The model highlights the critical necessity for cohesive approaches that enhance fertilizer application, boost nitrogen efficacy, and reduce climate effects to maintain agricultural productivity. This study connects mathematical theory to real-world data, enhancing the understanding of how climate change and farming practices together influence future food security, while establishing a foundation for creating adaptive policies that reconcile productivity with environmental sustainability.

-END-

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